A Meta-Heuristic Ant Colony Optimization Method for Solving Portfolio Optimization

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Abstract

This paper proposes a meta-heuristic ant colony optimization method to solve the portfolio optimization problem. The objective is to maximize the portfolio return and to minimize the portfolio risk simultaneously. To do so, the optimization problem is formulated as a total cost function to be minimized. The computational results on an example of five (05) stock portfolios not only show that the proposed method is capable and effective in finding the optimum portfolio return with the minimum of risk but also provides better solutions than another meta-heuristics based on genetic algorithms with respect to convergence time and efficiency.

Keywords: Meta-heuristics Optimization Methods, Ant Colony Optimization Method, Multi-objective, Portfolio Optimization.

Introduction

In finance, portfolio optimization is a problem that is still generating a great deal of attention. Clearly, the portfolio problem can be thought of as a multi-objective problem where two conflicting and dependent objectives are to be considered simultaneously: maximization of portfolio return and minimization of the portfolio risk. In an attempt to remedy this problem, *Markowitz* (1952) developed a quantitative model that has become known as the mean-variance model. The mean-variance model of *Markowitz* was the first mathematical formulation of the portfolio optimization problem. In this model, Harry *Markowitz* presented the portfolio optimization problem as a unique objective and linear function and dealt with it as a tradeoff problem between maximization of the portfolio expected return and minimization of the portfolio risk. Though the *Markowitz's* model constitutes a basis upon which other methods are developed, in practice however, the model is seen far from being satisfactory due to its neglect to account for constraints like maximum sizes of portfolio, minimum lots, transaction costs, preferences over assets, management costs and so on (Di Tollo and Rolli, 2008). Thus, there was a need to resort to other alternative models capable to include these constraints. As a result, a variety of exact optimization methods have been developed and applied in an attempt to solve portfolio optimization problem. Exact optimization methods are based on linear assumption and are therefore good for quadratic objective functions (deterministic) with a single objective (Sefiane and Benbouziane, 2012). Hence, when the objective is to maximize the portfolio return and to minimize the portfolio risk simultaneously subject to a set of constraints, the problem becomes a multi-objective problem for which these exact optimization methods cannot be useful. This situation has justified the search for other methods capable of dealing with more complex cases.

Recently meta-heuristics methods as such ant colony algorithms have been proposed and applied for non-linear and non-convex multi-objective optimization problems. They have been applied to portfolio optimization problems that also allow for side conditions like transaction cost, taxes, cardinality constraints, etc. Meta-heuristics optimization methods are flexible enough to optimize portfolios based on complex objective functions

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without the need for linear approximation of the objective function (Lyra, 2010).

Although it is fairly obvious that the main reason for the use of meta-heuristics in any problem as compared to exact methods is their advantage in terms of execution time, while accuracy favours exact methods, the aim of this paper however, is not to compare the proposed meta-heuristic ACO method with any of the exact methods. Suffice it to say that meta-heuristic methods are preferable to exact methods with respect to execution time (Blum and Roli, 2003), which should justify their use.

On the other hand, despite their success application in different fields as optimization methods, little search has been done to apply the Ant Colony Optimization (ACO) method in portfolio optimization problems. Hence, this paper aims to add to the existing searches in this area by presenting an example application of the ant colony optimization method to the financial portfolio optimization problem.

Optimization with Ant Colonies Algorithms

The ant colony algorithm which was firstly proposed by Dorigo and his collaborators, (Dorigo *et al*., 1991) is a meta-heuristic method that is used to optimize combinatorial problems (Dorigo and Gamberdalla, 1997; Dorigo *et al*., 1999; Dorigo and Blum, 2005). The ant colony optimization method is inspired from real ants' behaviour when searching for food. Information about food source is communicated between ants through individual substances, called pheromones. If an ant discovers the shortest route to the food source, it returns to the nest, leaving in its route a pheromone that allows other ants to follow the same route. These ants in turn will deposit more pheromones. By this stigmergy mechanism, ants are able to map out the shortest route between the nest and food sources.

Working Procedures of the Ant Colony Optimization Algorithm

The approach of ACO represents a problem of optimization on a graph whose nodes represent the decision variables of the optimization problem. Ants build solutions step by step while moving on the construction graph according to a stochastic construction procedure that depends on the use of pheromone trails and heuristic information. In the process of construction of the graph, each ant selects its move to a node according to information stored in the graph that is the amount of pheromone or the heuristic value which indicates the cost of local moves. The relative importance of these two pieces of information depends on the control parameters in the transition rule. Once a route is completed, information on the quality of this route will be modified. The ACO approach attempts to solve an optimization problem by repeating the following three steps Yu Ma-Tai (2007):

Step 1**: **Initialization of a Quantity of Pheromone: the quantity of pheromones on the components of the graph represents the attractiveness trail to guide ants to find better solutions in the search space. Most of ACO algorithms initialize components of the graph with a positive and small quantity bounded between 0 and 1. However, a feasible solution provided by another algorithm can also be used to contribute in promoting certain routes to go at the beginning of the iteration.

Step 2**: **Construction of the Solution by a Transition Rule Based: The ACO algorithm uses information from pheromone trails to find the optimal solution in a progressive manner. This mechanism comes, on one hand, from local information that provides current information on the environment, and on the other hand, from global information based on pheromones trail updating in favour of some paths. The process of construction of solution, known as construction of the graph, consists of traversing step by step the graph (arcs or nodes) through a stochastic transition rule. The probability of choice on emergent arcs or nodes depends on the amount of pheromones and local information about the emergent arc or node. At each stage of the construction of the graph, the constraints of the problem must be met. The transition rules must be adapted to each application problem.

The ant algorithm system is the first version of the algorithm of ACO. As indicated before, AS algorithm uses a transition rule to find a solution. The transition rule is to drive ants in the direction of the best solution. For an ant ***m*** located on the node **r**, the probability of choosing the emergent arc (r, ***s***) at iteration ***t*** is given by the following equation:

$$p_{rs}^{m}(t) = \begin{cases} \frac{[\tau_{rs}(t)]^{\alpha}\ [\eta_{rs}]^{\beta}}{\sum_{u\in J_r^m}([\tau_{ru}(t)]^{\alpha}\ [\eta_{ru}]^{\beta})} & if\ s \in J_r^m \\ 0 & or\ else \end{cases} \quad \text{(Eq.1)}$$

where

α β : are two parameters which determine the relative influence of pheromone trail and heuristic information.

$\tau_{rs}(t)$: Quantity of pheromone deposited on the arc (r, s) at iteration t

η_{rs} : a heuristic value, called visibility, defined by the inverse length of the arc (**r, s**) or the inverse cost of the arc (**r, s**)

J_n^m : set of nodes that remain to be visited by ant m positioned on node r.

The transition rule gives the probability with which an ant m chooses to move to the next node (i.e., from node r to node s) according to both local heuristic value and amount of pheromones.

The parameters α, β have the following influence on the algorithm behaviour. If $\alpha = 0$, ants choose nodes having better heuristic values. If however, $\beta = 0$, the probability of choice only depends on the amount of pheromones. When an ant completes a tour, the amount of pheromone is added proportionally to the quality of the traversed path.

Algorithm of Ant Colony System (ACS) mainly introduced a policy of balance between exploration and exploitation. The ACS defines a parameter $q_0 \in [0, 1]$. This q_0 serves as a criterion promoting the choice of emergent arcs or next nodes by exploitation or exploration. The transition rule is given by the following equation:

$$s = \begin{cases} \arg\max_{u\varepsilon J_r^m} \left\{\tau_{ru}(t)[\eta_{ru}]^{\beta}\right\} & si\ q \le qo\ \text{(exploitation)} \\ s & si\ q > qo\ \text{(exploration)} \end{cases} \quad \text{(Eq. 2)}$$

where:

S is the node chosen according to the equation (Eq.1).

$q \in [0, 1]$ is a random variable.

This transition rule allows to adjust the exploration if $q > q_0$ and intensification if $q \le q_0$.

Step 3: Updating Pheromone Trails: update consists of two rules: the local updating rule and the global updating rule. In the local updating rule the ant can deposit the pheromone immediately at every construction step. Global updating rule uses a factor of evaporation rate $\rho \in [0, 1]$ to reduce the density of pheromones on all paths found. The aim is to encourage exploration and to avoid stagnation of solutions in a local optimum. The update of the trails of pheromones is given by equation (2) as follows:

$$\tau_{rs}(t+1) = (1-\rho)\,\tau_{rs}(t) + \rho\sum_{m=1}^{m}\left(\Delta\tau_{rs}^{m}(t)\right) \quad \text{(Eq.3)}$$

where

$\tau_{rs}(t)$: the quantity of pheromones deposited on the arc (r, s) at iteration t;

$\Delta\tau_{rs}^{m}(t)$: the Quantity of pheromone deposited by m^{th} ant

ρ : is the pheromone evaporation coefficient.

M: is the number of ants

The update rule in the ACO depends on the problem application. For example, AS rank method (Bullnheimer *et al.*, 1997) reinforces the roads taking into account the best ants ranking. Max-Min AS (Stutzle. T. and Hoos, 1997) limits the amount of pheromone in the interval τmin and τmax to avoid the AS stagnation.

The theoretical problems considered in the ACO method are the algorithm convergence and the transition rule which can allow a more stable solution to the general combinatorial optimization problem, because the amount of pheromone plays a significant role in stimulating the performance of ACO. The realization of the same ACO algorithm gives different results and behaviours for two isomorphic problems (multiply an objective function of a problem by a constant) (Dehini and Sefiane, 2011).

For this, most ACO algorithms use the following update rule (Dorigo and Blum, 2005):

$$\tau_{rs}(t) = (1-\rho)\,\tau_{rs}(t-\rho) + \rho\sum_{m\varepsilon M^{upd}}^{m}\left(\frac{\Delta\tau_{rs}^{m}}{\sum_{m\varepsilon M^{upd}}\left(\Delta\tau_{rs}^{m}\right)}\right).\text{if } (r,s)\ \varepsilon\ p^{m}(t) \quad \text{(Eq.4)}$$

where:

M^{upd} : is the set of eligible solutions generated at iteration t.

ρ *:* is the rate of evaporation.

$\rho^m(t)$: is the path taken by the ant *m* at iteration *t*.

This formula is used to standardize the quantity pheromones added to avoid the influence of the objective order of the function.

The ACO Method and its Application to Portfolio Optimization Problems

In recent years, the meta-heuristic ant colony optimization method has been applied successfully to financial portfolio optimization problems. For instance, a recent study by Haqiqi and Kazemi (2012) applied the ACO method on a set of data from Tahran stock market for a four year period (2008-2011). In this study, the performance of ACO was compared with frontcon function of MATLAB software as an exact method. The results show that proposed ACO method is reliable but not preferred to an exact method. Furthermore, in an attempt to improve the algorithm performance, risk values obtained by ACO method, were compared with Lingo optimal results. While it was concluded that a significant difference between the risk values of ACO and Lingo optimal ones, the study argued that more attempts are needed to improve the ACO method toward a more applicable one. The study by Doerner *et al.* (2001) applied the ACO to multi-objective portfolio problems. The computational results via three approaches show that after the execution of 200 iterations ACO generates 110.3 efficient portfolios of the overall existing 138 efficient portfolios, whereas the use of a heuristic by itself only finds 97.9 portfolios on the average. The search for efficient portfolios on the basis of Monte Carlo Simulation does not prove successful even for a limited example, providing only 0.4 of the overall existing efficient portfolio on the average. Another indicator for the quality of ACO is the small percentage of all possible portfolios that the algorithm operates, e.g.; only 0.5% of the total search space is checked to find 128 out of 138 efficient portfolios in the version with 500 iterations. The results can be interpreted as good indications that ACO will generate satisfying results for problems that are too large to be enumerated completely. Another study by Doerner *et al.* (2002) applied a heuristic approach named Pareto Ant Colony Algorithm to a portfolio selection problem. A comparison of the Pareto-ACO to other two approaches that are Pareto Simulated Annealing and Non-Dominated Sorting Genetic Algorithm concludes that Pareto-ACO is the most efficient one. It was concluded also that the applications of the P-ACO to the project portfolio selection problem under considerations has three advantages: (1) it can handle the (complex) project interactions and constraints better than the other two heuristics, (2) it is robust in that it shows very good results on various problem characteristics (e.g., many of few constraints and/or objectives), and (3) heuristic information can easily be plugged into the algorithm.

Mathematical Formulation of the Problem

The portfolio optimization problem can be mathematically described as following:

The expected return of the individual assets ***i*** is presented as a polynomial of first degree:

$$E(w_i) = w_i \cdot r_i \tag{1}$$

where

w_i denotes the weight of the individual asset ***i***.

r_i denotes the expected return of asset i.

Thus the total expected return of portfolio P can be written as: $Rp = \sum_{i=1}^{n} E(w_i)$,

where:

n is the number of assets.

The objective function of the portfolio return to be maximized can be written as follows:

$$\max \left\{ Rp = \sum_{i=1}^{n} E(w_i) \right\} \tag{2}$$

Under the following constraints:

$$\sum_{i=1}^{n} w_i = 1 \tag{3}$$

$$W_i^{\min} \le w_i \le W_i^{\max} \tag{4}$$

where:

$W_i^{\max}$ and $W_i^{\min}$: maximum and minimum weights of asset ***i***.

For a positive portfolio return (what so ever are the weights values), let

$$\sum_{i=1}^{n} r_i w_i \geq 0 \tag{5}$$

The objective function of the Portfolio variance (Portfolio risk) to minimize is presented as a polynomial of second degree:

$$\sigma^2 p = \sum_{i=1}^{n} (w_i^2 \sigma^2(r_i)) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} 2 w_i w_j \operatorname{cov}(r_i, r_j) \tag{6}$$

$\sigma^2(r_i)$: Variance of asset ***i***

$\operatorname{cov}(r_i, r_j)$: Covariance between asset ***i*** and asset ***j***

Application of the ACO method to the Portfolio Problem

With the ACO method the basic idea is to model the problem to solve as the search for the minimum cost (shortest path) that meets the problems' restrictions (Armananzas and Lozano, 2005). Hence, the objective function that provides the least value (i.e. minimum total function) should lead to the best solution.

Application of the ACO method to the portfolio optimization is linked to the selection of weights of portfolio invested in each asset as to maximize the portfolio return and minimize the portfolio risk simultaneously. Hence to meet this condition, the multi-objective function to minimize is presented in Eq. 7 below as an algebraic summation of the portfolio return and the portfolio variance, where the least value of the objective function defines the best solution (i.e., best portfolio). In addition to this, a penalty function method is used to handle the constraints (Eq. 10).

The multi objective function, which is the total function to minimize therefore, can be expressed as follows:

$$TFp = Rp - \sigma^2 p \tag{7}$$

For the application of the ACO application, a penalty function method is used to handle the constraints.

$$g_i(w_i) \geq 0 \qquad i = 0, \ldots\ldots. n \tag{8}$$

These are the inequality constraints type

$$h_j(w_i) = 0 \qquad j = 0, \ldots\ldots n \tag{9}$$

The problem is transformed into a penalty function, which is presented as follows:

$$F(w_i, r_k) = TFp + \frac{1}{\mu_k} \sum_{j=1}^{n} [h_j(w_i)]^2 + \mu_k \sum_{j=1}^{m} [\frac{1}{g_i(w_i)}]^2 \tag{10}$$

where μ_k is the penalty coefficient.

The Example Application

Although no detailed comparison between our proposed ACO method and genetic algorithm is sought in this paper, the same applied data on genetic algorithm method in a previous study by Sefiane and Benbouziane (2012) is reused and applied to the ACO method in the present paper. The purpose is to show the difference between the results obtained under each method and draw a conclusion about which of the two methods is more efficient than the other. The data consist of five (05) stock portfolio for a period of five years**.**

The portfolio average return and the portfolio variance are given in Table 1.

Table 1: Portfolio Average Return and Portfolio Variance

Year	*Stock 1*	*Stock 2*	*Stock 3*	*Stock 4*	*Stock 5*
2007	-0.15	0.29	0.38	0.18	-0.10
2008	0.05	0.18	0.63	-0.12	0.15
2009	-0.43	0.24	0.46	0.42	0.15
2010	0.79	0.25	0.36	0.24	0.10
2011	0.32	0.17	-0.57	0.30	0.25

The mean return for each asset and the covariance matrix are given in Table 2 and 3.

Table 2: The Mean Returns for Each Asset

	Stock 1	*Stock 2*	*Stock 3*	*Stock 4*	*Stock 5*
Mean return r_i	0,116	0,226	0,252	0,204	0,11

The shaded cells represent the variance of asset ***i*** ($\sigma^2(r_i)$)

For the purpose of the application of the ACO, the initial value parameters shown in Table 4 were utilized:

Table 3: The Covariance Matrix

	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5
Stock 1	0,21728	-0,003376	-0,053492	-0,009264	0,01064
Stock 2	-0,003376	0,00253	0,008468	0,002376	-0,00456
Stock 3	-0,053492	0,008468	0,22247	-0,031128	-0,02392
Stock 4	-0,009264	0,002376	-0,031128	0,04068	0,00276
Stock 5	0,01064	-0,00456	-0,02392	0,00276	0,01675

Table 4: Initial Value Parameters of ACO

Ant number 10
Maximum cycle Time 150
Initial value of pheromone quantity $\tau_0 = 0.1$
the pheromone evaporation coefficient $\rho = 0.35$
Relative important parameter of trail intensity $\alpha = 1.1$
Relative important parameter of visibility $\beta = 1.5$

The ACO Computational Results

The test was conducted using the MATLAB software package. The computational results show that after the execution of 150 iterations with a number of 10 ants, ACO generates the following results:

Portfolio weights	The computational results
W1: 0.0471113112170	The optimum portfolio return: 0.222954811549638
W2: 0.4156542709600	The portfolio variance (optimum risk): 0.0311
W3: 0.3898056443999	The least value: 4.4863167339614
W4: 0.0892984731830	Execution time: 4.0142 sec
W5: 0.0645829390309	

Figure 1 shows the convergence of the algorithm results as regards the minimum total function (i.e., the least value), and Figure 2 shows the allocation of weights of portfolio invested in each asset as to maximize the portfolio return and minimize the portfolio risk.

Comparison of the computational results between the ACO method and the GA method

ACO	GA
The portfolio return: 0.222954811549638	The portfolio return: 0.2213
The portfolio variance (risk): 0.0311	The portfolio variance (risk): 0.0801
The least value: 4.4863167339614	The least value: 4.5984
Execution time: 4.0142 sec.	Execution time: 4.0470 sec.

Figure 1: Objective Function Values for Number of Iterations

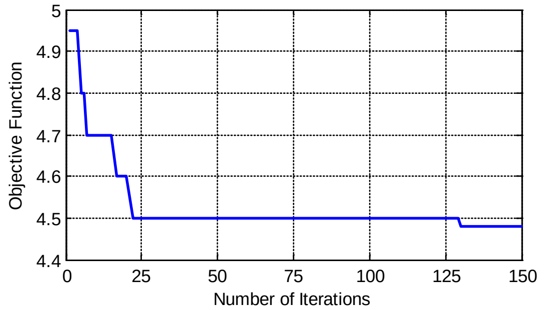
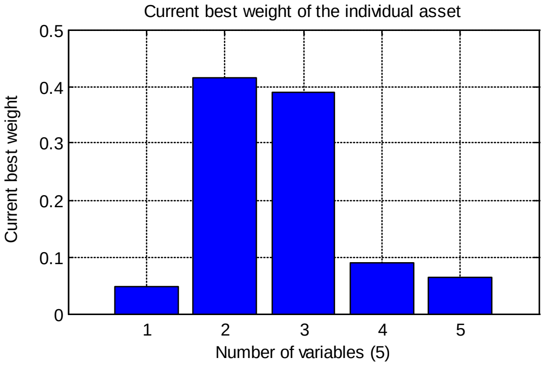


Figure 2: Weights for the Optimal Portfolio for Each Asset



Conclusion and Recommendations

The main objective of this paper is to illustrate via a five (05) asset portfolio example the capability of the meta-heuristic ACO method in solving portfolio optimisation problems. In order to achieve this goal, the objective function was formulated as to maximize the portfolio return and to minimize the portfolio risk simultaneously, and consequently the least value of the objective function (minimum total cost) should lead to the best solution (i.e., optimal portfolio). The ACO method was able to identify the optimal portfolio return with the minimum of risk.

With the understanding that the genetic algorithm does not work well for all the problems (Zhou, 2006), it's not surprising that when comparing the results obtained under the ant colony optimization method and the results obtained under the genetic algorithm method, our proposed approach ant colony optimisation method is better in terms of both efficiency and execution time, where the portfolio return rate was found to be 22.29% and slightly inferior under the GA with 22,13%. As for the portfolio variance rate (i.e., risk) it was found to be only 3.11% under the ACO algorithm and 8% under the GA algorithm. In addition, under the ant colony optimization method the execution time (i.e., converging toward the optimal solution) was found to be only 4.0142 sec. which is less than that of the genetic algorithm which is 4.0470 sec. In the light of the computational results, it is concluded that the proposed ACO method is not only capable and effective in finding the optimal portfolio but it also gives better results than the GA in less time.

Although this paper has added to the existing searches about the capability and efficiency of the ACO method in solving portfolio optimization problems, limitations still exist. First, it should be observed that the number of ants is limited to 10 and the number of iterations to 150, therefore it is recommended that future searches should study for an increased number of ants and iterations which will have a better effect on the ACO performance. Second, it should be observed that our proposed method was tested on a simple example of five (05) asset portfolio; future searches are recommended to consider if with a larger number of asserts the same conclusion will always hold true. Third, a detailed comparison between the ACO method and the GA one is left to future searches to draw a precise judgement about which of the two methods is better in terms of accuracy, efficiency, simplicity or any other reasonable criterion.

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