Article can be accessed online at http://www.publishingindia.com

A Common Fixed Point Theorem Using Implicit Relation in Fuzzy Metric Space

Saurabh Manro*

Abstract

In this paper, we prove a common fixed point theorem in fuzzy metric space by combining the ideas of pointwiseR- weak commutativity and reciprocal continuity of mappings satisfying contractive conditions with an implicit relation. Our results improve and extend many known results in the literature. Moreover, example and application are given to illustrate the usability of the obtained results.

Keywords: Fuzzy Metric Space, Reciprocal Continuity, R-Weakly Commuting Mappings, Implicit Relation.

1. Introduction

In 1965, Zadehintroduced the concept of fuzzy set as a new way to represent vagueness in our everyday life. However, when the uncertainty is due to fuzziness rather than randomness, as sometimes in the measurement of an ordinary length, it seems that the concept of a fuzzy metric space is more suitable. We can divide thisinto two groups. The first group involves those results in which a fuzzy metric on a set X is treated as a map where X represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distances are set up between fuzzy objects. On the other hand in second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy. Kramosil*et al*. (1975) have introduced the concept of fuzzy metric spaces in different ways (for detail, one can see [1-11]). In this paper, we prove a common fixed point theorem in fuzzy metric space by combining the ideas of pointwise*R*- weak commutativity and reciprocal continuity of mappings satisfying contractive conditions with an implicit relation.

2. Preliminaries

The concept of triangular norms (t-norms) is originally introduced by Menger in study of statistical metric spaces.

Definition 2.1 (Schweizer&Sklar, 1983)A binary operation * : [0,1]×[0,1] ♦ [0,1] is continuous t-norm if *

satisfies the following conditions:

(i) * is commutative and associative;

(ii) * is continuous;

(iii) $a* 1 = a$ for all aÎ[0,1] ;

(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all a,b,c,dÎ[0,1] .

Examples of t-norms are: $a*b = min\{a, b\}$, $a*b = ab$ and $a*b = max\{a+b-1,0\}$.

Kramosil et al. [3] introduced the concept of fuzzy metric spaces as follows:

Definition 2.2 (Kramosil&Michalek, 1975)A 3-tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an

arbitrary set, * is a continuous t-norm, and M is fuzzy sets on $X2\times[0, \infty)$ satisfying the

following conditions for all x, y, z└ X and $s, t > 0$

(i) $M(x, y, 0) = 0$;

(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

---

* Researcher, School of Mathematics and Computer Application, Thapar University, Patiala, Punjab, India. Email: sauravmanro@hotmail.com

(iii) $M(x, y, t) = M(y, x, t)$;

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(v) $M(x, y, .) : [0, \infty) \rightarrow [0, 1]$ is left continuous.

Then $(X,M,*)$ is called a fuzzy metric space on X. The function $M(x, y, t)$ denote the

degree of nearness between x and y w.r.t. t respectively.

Remark 2.3 (Kramosil&Michalek, 1975)In fuzzy metric space $(X,M,*)$, $M(x, y, .)$ is non-decreasing for all

x, yÎX.

Definition 2.4 (Kramosil&Michalek, 1975)Let $(X,M,*)$ be a fuzzy metric space. Then a sequence $\{x_n\}$ in X is

said to be

(a) convergent to a point *xÎX*if, for all $t > 0$,

$\lim_{n\to\infty} M(x_n, x, t) = 1$.

(b) Cauchy sequence if, for all $t > 0$ and $p > 0$,

$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$.

Definition 2.5 (Kramosil&Michalek, 1975)A fuzzy metric space $(X,M,*)$ is said to be complete if and only if

every Cauchy sequence in X is convergent.

Definition 2.6(Vasuki, 1999)A pair of self mappings (A , S) of a fuzzy metric space $(X,M,*)$ is

said to be commuting if $M(ASx , SAx , t) = 1$ for all *xÎX*.

Definition 2.7(Vasuki, 1999)A pair of self mappings (A , S) of a fuzzy metric space $(X,M,*)$ is

said to be weakly commuting if $M(ASx , SAx , t) \geq M(Ax, Sx, t)$ for all *xÎX*and $t > 0$.

Definition 2.8(Jungck& Rhoades, 2006) A pair of self-mappings (A, S) of a fuzzy metric space $(X,M,*)$ is

said to be compatible if $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a

sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u$ for some *uÎX*.

Definition 2.9(Jungck& Rhoades, 2006)Let $(X,M,*)$ be a fuzzy metric space,A and S be self-maps on X. A point*xÎX*is called a coincidence point of A and S iff$Ax = Sx$. In this case, $w = Ax = Sx$

is called a point of coincidence of A and S.

Definition 2.10(Jungck& Rhoades, 2006)A pair of self mappings (A, S) of a fuzzy metric space $(X,M,*)$ is

said to be weakly compatible if they commute at the coincidence points i.e., if $Au = Su$

for some *u*⌊ *X* , then $ASu = SAu$.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 2.11(Vasuki, 1999)A pair of self mappings (A, S) of a fuzzy metric space $(X,M,*)$ is

said to be pointwise$R$-weakly commuting if given *xÎX*, there exist $R > 0$ such that

(, ,) , ,

t

M ASxSAx t M AxSx

R

—

≥| √

⎝ ↲

for all $t > 0$.

Clearly, every pair of weakly commuting mappings is pointwise$R$-weakly commuting with$R = 1$.

Definition 2.12(Pant, 1999) Two mappings A and S of a fuzzy metric space $(X,M,*)$ will be called reciprocally continuous if , *n nASu*♦ *AzSAu*®*Sz*, whenever $\{u_n\}$ is a sequence

such that , *n nAu* ♦ *z Su* ♦ *z* for some *z*⌊ *X*.

If A and S are both continuous, then they are obviously reciprocally continuous but converse is not true.

Lemma 2.1(Kramosil&Michalek, 1975)Let $\{u_n\}$ is a sequence in a fuzzy metric space $(X,M,*)$. If there exists a constant*h*⌊(0,1) such that

1 1 (, ,) (, ,), n nnn*M u uht M u u t* + − ≥ *n* = 1, 2, 3, …

Then $\{u_n\}$ is a Cauchy sequence in X.

3. Main Result

In our result, we deal with the class φ ⌊Φ of all real continuous functions

( )4

φ : *R* R_+ ♦ , non-decreasing in the first argument and satisfying the following

conditions:

(1.1) for $u, v \geq 0$, $\varphi(u,v,u,v) \geq 0$ or $\varphi(u,v,v,u) \geq 0$Þ$u \geq v$

(1.2) $\varphi(u,u,1,1) \geq 0$ for all $u \geq 1$,

Example 3.1: Define 1 2 3 4 1 2 3 4 φ(*t* , *t* , *t* , *t*) =14*t* -12*t* + 6*t* –8*t* . Then φ ⌊Φ.

Lemma 3.1: Let $(X,M,*)$ be a complete fuzzy metric space. Further, let (A, S) and (B, T)

bepointwise*R*-weakly commuting pairs of self mappings of X satisfying

(2.1) *A*(*X*) Í*T*(*X*),*B*(*X*)⎸ *S*(*X*) ,

(2.2) $\varphi\big(M(Au,Bv,ht),M(Su,Tv,t),M(Au,Su,t),M(Bv,Tv,ht)\big) \geq 0$;

(2.3) $\varphi\big(M(Au,Bv,ht),M(Su,Tv,t),M(Au,Su,ht),M(Bv,Tv,t)\big) \geq 0$;

for all *u*, *v* ⌊ *X*, *t* > 0 , *h*⌊(0,1) and some φ ⌊Φ. Then the continuity of one of the

mappings in compatible pair (A, S) or (B, T) on $(X,M,*)$ implies their reciprocal

continuity.

Proof: First, assume that *A* and *S* are compatible and *S* is continuous. We show that *A*

and*S* are reciprocally continuous. Let $\{u_n\}$ be a sequence such that $Au_n \to z$ and $Su_n \to z$

for some *z* in *X* as *n*→× . Since *S* is continuous, we have $SAu_n \to Sz$and $SSu_n \to Sz$as

n→× and since (*A*, *S*) is compatible, we have

lim (, ,) 1

lim (, ,) 1

nn

n

n

n

M ASuSAu t

M ASuSz t

♦×

♦×

=

⇒ =

That is $ASu_n \to Sz$as *n*→× . By (2.1), for each *n*, there exists $v_n$in *X* such that $ASu_n = Tv_n$.

Thus, we have $SSu_n \to Sz$, $SAu_n \to Sz$, $ASu_n \to Sz$and $Tv_n \to Sz$as *n*→× whenever $ASu_n =$

Tv_n.

Now we claim that $Bv_n \to Sz$as *n*→× .

Suppose not, then by (2.2)

((, ,), (, ,), (, ,), (, ,)) 0; n nnnnnnnφ *M ASuBvht M SSuTv t M ASuSSu t M BvTvht*≥

Taking *n*→× ,

( )

( )

(, ,), (, ,), (, ,), (, ,) 0;

(, ,),1,1, (, ,) 0;

nn

nn

M SzBvht M SzSz t M SzSz t M BvSzht

M SzBvht M BvSzht

φ

φ

≥

≥

By (1.1), we have

(, ,) 1 *n M SzBvht*≥ for all $t > 0$ as n→× .

Hence (, ,) 1 *n M SzBvht*= as n→× .

Thus, Bv_n→Szas n→× .

Again by (2.2),

( )

( )

(, ,), (, ,), (, ,), (, ,) 0;

(, ,),1, (, ,),1 0

M AzSzht M SzSz t M AzSz t M SzSzht

M AzSzht M AzSz t

φ

φ

≥

≥

As φ is non decreasing in first argument,

$\varphi\left(M(Az,Sz,t),1,M(Az,Sz,t),1\right) \geq 0$

By (1.1)

$M(Az, Sz, t)$ ³1 for all $t > 0$. This gives

$M(Az, Sz, t) = 1$. Thus $Az = Sz$.

Therefore, SAu_n→Sz, ASu_n→$Sz = Az$as n→× .

Hence, A and$S$ are reciprocally continuous on X. If the pair (B, T) is assumed to be

compatible and T is continuous, the proof is similar.

Theorem 3.1: Let $(X,M,*)$ be a complete fuzzy metric space. Further, let (A, S) and

(B, T) be pointwise$R$- weakly commuting pairs of self mappings of X satisfying (2.1),

(2.2), (2.3).

If one of the mappings in compatible pair (A, S) or (B, T) is continuous, then A, B, S and

T have a unique common fixed point.

Proof: let u0⌊ X. By (2.1), we define the sequences $\{u_n\}$ and $\{v_n\}$ in X such that for all

$n = 0, 1, 2, \ldots$

$v_{2n+1} = Au_{2n} = Tu_{2n+1}$, $v_{2n+2} = Bu_{2n+1} = Su_{2n+2}$.

By (2.2),

( )

( )

2 2 1 2 2 1 2 2 2 1 2 1

2 1 2 2 2 2 1 2 1 2 2 2 2 1

(, ,), (, ,), (, ,), (, ,) 0;

(, ,), (, ,), (, ,), (, ,) 0;

nnnnnnnn

nnnnnnnn

M Au Bu ht M Su Tu t M Au Su t M Bu Tuht

M v vht M v v t M v v t M v vht

φ

φ

+ + + +

+ + + + + +

≥

≥

By (1.1),

2 1 2 2 2 2 1 (, ,) (, ,) *n nnnM v vht M v v t* + + + ≥

Similarly, by (2.3), we have

2 2 2 3 2 1 2 2 (, ,) (, ,) *n nnnM v vht M v v t* + + + + ≥

Therefore, for any n and t, we have

1 1 (, ,) (, ,) *n nnnM v vht M v v t* + − ≥

Hence, by Lemma 3.1, $\{v_n\}$ is a Cauchy sequence in *X*. Since *X* is

complete,$\{v_n\}$converges to *z*⌊ *X*. Its subsequences $\{Au_{2n}\}$, $\{Tu_{2n+1}\}$, $\{Bu_{2n+1}\}$ and

$\{Su_{2n+2}\}$ also converges to *z*.

Now, suppose that (*A*, *S*) is a compatible pair and *S* is continuous. Then by above lemma,

A and*S* are reciprocally continuous, then $SAu_n \to Sz$, $ASu_n \to Az$as *n*→× .

As, (*A*, *S*) is a compatible pair. This implies

lim (, ,) 1;

(, ,) 1

nn

n

M ASuSAu t

M AzSz t

♦×

=

=

Hence, *Az* = *Sz*.

Since *A*(*X*) Í*T*(*X*) , there exists a point *p*⌊ *X* such that *Az* = *Tp* = *Sz*.

By (2.2),

( )

( )

( )

(, ,), (, ,), (, ,), (, ,) 0;

(, ,), (, ,), (, ,), (, ,) 0;

(, ,),1,1, (, ,) 0;

M AzBpht M SzTp t M AzSz t M BpTpht

M AzBpht M SzSz t M AzAz t M BpAzht

M AzBpht M BpAzht

φ

φ

φ

≥

≥

≥

By (1.1), we get

M(*Az*,*Bp*,*ht*) ³1 for all $t > 0$.

This gives, *M*(*Az*,*Bp*,*ht*) =1

Hence, *Az* = *Bp*

Thus, *Az* = *Bp* =*Sz* = *Tp*

Since, *A* and *S* are pointwise*R* – weakly commuting mappings, there exists $R > 0$, such

that(, ,) , , 1

t

M ASzSAz t M AzSz

R

—

≥| √ =

⎝ ↵

.

Therefore, *ASz* = *SAz*and *AAz* = *ASz* = *SAz* = *SSz*.

Similarly, *B* and *T* are pointwise*R*-weakly commuting mappings, we have *BBp* = *BTp* =

TBp = *TTp*.

Again by (2.2),

$\varphi\left(M(AAz,Bp,ht),M(SAz,Tp,t),M(AAz,SAz,t),M(Bp,Tp,ht)\right) \geq 0;$

$\varphi\left(M(AAz, Az,ht),M(AAz, Az,t),1,1\right) \geq 0;$

As φ is non decreasing in first argument,

$\varphi\big(M(AAz, Az,t),M(AAz, Az,t),1,1\big) \geq 0;$

By (1.2)

$M(AAz, Az, t)$ ${}^{3}1$ for all $t > 0$. This gives $M(AAz,Az,t) = 1$

Implies $AAz = Az = SAz$. Hence $Az$is common fixed point of A and S. Similarly by (2.2),

$Bp = Az$is a common fixed point of B and T. Hence, $Az$is a common fixed point of A, B,

S and T.

For Uniqueness: Suppose that Ap $(\neq Az)$ is another common fixed point of A, B, S and T.

Then by (2.2),

( )

( )

( )

( )

(, ,), (, ,), (, ,), (, ,) 0;

(, ,), (, ,), (, ,), (, ,) 0;

(, ,), (, ,),1,1 0;

(, ,), (, ,),1,1 0;

M AAzBApht M SAzTAp t M AAzSAz t M BApTApht

M AzApht M AzAp t M AzAz t M ApApht

M AzApht M AzAp t

M AzAp t M AzAp t

φ

φ

φ

φ

≥

≥

≥

≥

By (1.2),

$M(Az,Ap,t) \geq 1$ for all $t > 0$

Hence $Az = Ap$

Thus, uniqueness follows.

Taking $S = T = Ix$ in above theorem, we get following result:

Corollary 3.1: Let $(X,M,*)$ be a complete fuzzy metric space. Further, let A and B are

reciprocally continuous mappings on X satisfying

(2.4) $\varphi\big(M(Au,Bv,ht),M(u,v,t),M(Au,u,t),M(Bv,v,ht)\big) \geq 0;$

(2.5) $\varphi\big(M(Au,Bv,ht),M(u,v,t),M(Au,u,ht),M(Bv,v,t)\big) \geq 0;$

for all u , v in X, $t > 0$ and h⎣(0,1) and some φ ⎣Φ then pairs A and B has a unique

common fixed point.

Example 3.2: let $X = R_+$ and let M be defined by (, ,)

t

M u v t

t u v

=

+ −

. Then $(X,M,*)$

is complete fuzzy metric space. Let A, B, S and T be self mappings of X defined as

$A(0) = 0$, $Au = 1$ if $u > 0$,

$B(u) = 0$ *if* $u = 0$ or $u > 6$, $Bu = 2$ if $0 < u \leq 6$,

$S(0) = 0$, $S(u) = 2$ if $u > 0$,

$T(0) = 0$, $T(u) = 4$ if $0 < u \leq 6$, $T(u) = u - 6$ if $u > 6$.

Then A, B, S and T satisfy all the conditions of above theorem with h⎣(0,1) and have a

unique common fixed point $u = 0$.

References

Imdad, M., & Ali, J.(2008).Jungck's common fixed point theorem and *E.A*Property.*ActaMathematicaSinica*, 24(1), 87-94.

Jungck, G., & Rhoades, B.E. (2006). Fixed point theorems for occasionallyweakly compatible mappings. *Fixed Point Theory*, 7(2), 286-296.

Kramosil, I., &Michalek, J.(1975).Fuzzy metric and statistical metric spaces. *Kybernetica*, 11(5), 326-334.

Kumar, S., & Pant, B.D. (2008).Faculty of sciences and mathematics.*Universityof Nis Serbia*, 22(2), 43-52.

Mishra, S.N., Sharma, N., & Singh, S.L.(1994). Common fixed points of maps on fuzzy metric spaces. *International Journal of Mathematics and Mathematical Sciences*, 17(2), 253-258.

Pant, R.P.(1994). Common fixed points of non commuting mappings. *Journal of Mathematical Analysis and Applications*, 188(2), 436-440.

Pant, R.P.(1999). A common fixed point theorem under a new condition. *Indian Journal of Pure and Applied Mathematics*,30(2), 147-152.

Pathak, H.K., Cho, Y. J., &Kang, S.M.(1997). Remarks on R-Weakly commuting mappings and common fixed point theorems.*Bulletin of the Korean Mathematical Society*, 34(2), 247-257.

Schweizer, B., &Sklar, A.(1983). Probabilistic metric spaces.*North HollandAmsterdam*.

Vasuki, R.(1999). Common fixed points for R-weakly commuting maps in fuzzy metric spaces.*Indian Journal of Pure and Applied Mathematics*,30(4), 419-423.

Zadeh, L. (1965). Fuzzy sets.*Information and control*, 89(3), 338-353.