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On Volatility Trading & Option Greeks

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Abstract

Commensurate with this exponential growth in the depth and breadth of derivative markets and the range of financial products traded therein, there needs to be developed a comprehensive mathematical framework to support the, hitherto, empirically established features of trading strategies involving these instruments. It is the objective of this article, to provide a mathematical backup for the various properties of 'volatility trading' strategy using call options. Additionally, an attempt is made to elucidate the implications of behavior of various 'option Greeks' on volatility trading.

Keywords: Financial Derivatives, Trading Strategies, Option Greeks, Black Scholes Model, Volatility Trading

Introduction

A complete renaissance of the global financial markets has taken place in the last two decades. With the introduction of a multiplicity of tradable instruments, financial products of immense variety and possessing features compatible with the goals and needs of a large segment of the community are now available for trading. Popular awareness about the derivative instruments and their salient characteristics has increased manifold in the recent past. Use of these instruments by banks, corporates and individual investors as investment avenues has also escalated with this growing familiarity, thereby adding to the trading volumes in various financial markets.

Commensurate with this exponential growth in the depth and breadth of derivative markets and the range of financial products traded therein, there needs to be developed a comprehensive mathematical framework to support the, hitherto, empirically established features of these instruments. It is the objective of this article, to provide a mathematical backup for the various properties of financial derivative securities with special reference to trading of volatility.

Review of Literature

The Black Scholes valuation model (Black & Scholes, 1973; Black, 1976; Hull, 2012; Jarrow & Rudd, 1983; Merton, 1973) constitutes the cornerstone of modern finance. The model, as initially propounded, envisaged the formulation of a partial differential equation for the pricing of the European call option by creating a portfolio that exactly replicated the payoff of the option and the value of whose constituents was known. The European call option is a financial contingent claim that entails a right (but not an obligation) to the holder of the option to buy one unit of the underlying asset at a future date (called the exercise date or maturity date) at a price (called the exercise price). The Black Scholes formulation is premised on the following assumptions that are now, collectively referred to as the Black Scholes world (Merton, 1971a, 1973b; Black & Scholes, 1973; Jarrow & Rudd, 1983; Wilmott, 2000; McDonald, 2002; Hull, 2012) and are assumed to hold in the sequel:

1. The stock price follows the generalized Brownian motion process with constant mean and volatility.
2. The short selling of securities with full use of proceeds is permitted.
3. There are no transaction costs or taxes. All securities are perfectly divisible.
4. There are no dividends during the life of the derivative.
5. There are no riskless arbitrage opportunities.
6. Security trading is continuous.
7. The risk-free rate of interest, r, is constant and the same for all maturities.

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Option Greeks represent the sensitivities of the derivative prices to the various factors that influence such prices viz. price of the underlying, rate of change of price of underlying, time to decay, volatility and risk free rates. These are measured in terms of the partial derivatives of the Black Scholes formula with respect to these factors. Accordingly, we have option Greeks by the names of delta $\left(\Delta = \frac{\partial c}{\partial S}\right)$, gamma $\left(\Gamma = \frac{\partial^2 c}{\partial S^2}\right)$, theta $\left(\Theta = \frac{\partial c}{\partial t}\right)$, vega $\left(\upsilon = \frac{\partial c}{\partial \sigma}\right)$ and rho $\left(\rho = \frac{\partial c}{\partial r}\right)$ (Haug, 1997; Hull, 2012; McDonald, 2002; Taleb, 1997, Webb, 1999; Wilmott, 2000).

Volatility trading is a group of strategies that enables the strategeur to profit from the magnitude of price swings of the instruments rather than the direction of such swings i.e. the market player would profit from the quantum of price fluctuations (volatility) in an interval of time irrespective of whether such movements involve increases in price or decreases. A simple long volatility strategy comprises of a combination of taking long positions in call options and short positions in the underlying or taking long positions in put options and also in the underlying. Long volatility positions gain when realized volatility (over the lifetime of the option) is higher than implied volatility at inception. Since such strategies rely heavily on price movements for success, sensitivity analysis of prices with respect to various factors (represented by option Greek valuations) at various points become crucial for the player. This sets the backdrop and leads us to the theme of the current work.

Research Methodology

This is a mathematical paper aimed at providing mathematical backup for the various properties of 'volatility trading' strategy using call options. It also provides quantitative rationale for the implications of empirical behavior of various 'option Greeks' in volatility trading. Besides explaining the mathematics behind 'volatility trading' and the various features of this trading strategy on the basis of mathematical formulations, the article also explains quantitatively, the role of 'option Greeks' in this strategy.

Analysis & Results

Black Scholes Formulation of Volatility Trading

We start by considering the performance of a fund manager X that has a long position in a call option and a short position in $\Delta_i = \frac{\partial c_i}{\partial S_i}$ units of the underlying asset (say, stock for concreteness) at any instant of time t where c_t and S_t are respectively the unit instantaneous values of the call option and the underlying stock. This portfolio is usually termed as a "long volatility trade" in market parlance. Obviously the exposure of the fund manager at instant of time t is zero, as both assets are delta-equivalent at any instant. The value of the portfolio $V_t = c_t - \Delta S_t$ at any instant t, however, need not necessarily vanish.

Let us, now, assume that the stock price increases spontaneously from a given $S_{0,t}$ at $t = $ t to $S_{1,t}$ whence the value of X's portfolio changes by:

$$\begin{aligned}\pi\left(S_{0,t}, S_{1,t}, t\right) &= \pi_{long\ call}\left(S_{0,t}, S_{1,t}, t\right) \\ &\quad + \Delta_t \pi_{short\ stock}\left(S_{0,t}, S_{1,t}, t\right) \\ &= \left\{\begin{matrix}\left[S_{1,t} N\left(d_1^{1,t}\right) - Ke^{-r(T-t)} N\left(d_2^{1,t}\right)\right] \\ -\left[S_{0,t} N\left(d_1^{0,t}\right) - Ke^{-r(T-t)} N\left(d_2^{0,t}\right)\right]\end{matrix}\right\} \\ &\quad -\Delta_t\left(S_{1,t} - S_{0,t}\right)\end{aligned} \tag{1}$$

in the Black Scholes world. In eq. (1), we have used the standard notation (Hull, 2012):

$$d_1^{0,t} = \frac{\ln \frac{S_{0,t}}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}};$$

$$d_2^{0,t} = d_1^{0,t} - \sigma\sqrt{T-t};$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$ is the

standard normal cumulative probability distribution function. K is the exercise price of the option, σ, r are respectively the volatility of the underlying stock and the risk free rate of return. Similarly, $d_1^{1,t}$ and $d_2^{1,t}$ are also defined in terms of $S_{1,t}$.

Now, it can be shown that in the Black Scholes set-up (Appendix 1), $\Delta_t = \frac{\partial c_t}{\partial S_{0,t}} = N\left(d_1^{0,t}\right)$ whence, from eq. (1), we have,with $\Delta_t = N\left(d_1^{0,t}\right)$, that

$$\pi = \begin{Bmatrix} S_{1,t}\left[N\left(d_1^{1,t}\right) - N\left(d_1^{0,t}\right)\right] \\ -Ke^{-r(T-t)}\left[N\left(d_2^{1,t}\right) - N\left(d_2^{0,t}\right)\right] \end{Bmatrix}$$

$$= \begin{Bmatrix} \left[S_{1,t}N\left(d_1^{1,t}\right) - Ke^{-r(T-t)}N\left(d_2^{1,t}\right)\right] \\ -\left[S_{1,t}N\left(d_1^{0,t}\right) - Ke^{-r(T-t)}N\left(d_2^{0,t}\right)\right] \end{Bmatrix} \tag{2}$$

Towards ascertaining the extremals of π, we note that

$$\frac{\partial \pi}{\partial S_{1,t}} = N\left(d_1^{1,t}\right) - N\left(d_1^{0,t}\right) \text{ and} \tag{3}$$

$$\frac{\partial^2 \pi}{\partial S_{1,t}^2} = \frac{1}{\sigma S_{1,t}\sqrt{T-t}} N'\left(d_1^{1,t}\right)$$

$$= \frac{1}{\sigma S_{1,t}\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(d_1^{1,t}\right)^2}{2}} > 0 \tag{4}$$

for all values of $S_{1,t}$

Eqs. (3) & (4) show that the extremal of eq. (2) occurs at $S_{1,t} = S_{0,t}$ has the value $\pi_{\min} = 0$ and that the extremal is a minima. It follows that for any value of $S_{1,t} \neq S_{0,t}$ i.e. any spontaneous rise or fall in the value of stock, π attains a positive value and, further that it is upside unbounded showing conclusively that the call portfolio outperforms a stock portfolio of equal delta irrespective of the direction of the change in price of the underlying stock. There is another way of establishing this result. For this purpose, we write eq. (2) in the form

$$\pi\left(\delta S_t, \delta c_t, t\right) = \delta c_t - \Delta_t\left(\delta S_t\right) \tag{5}$$

where Δ_t is the slope of the call price curve at $S_{0,t}$. The former term will exceed the latter If there is an increase in stock price, since the call price curve increases in steepness (slope) as the stock price rises and vice versa. In the Black Scholes Model, the slope $\Delta_t = \frac{\partial c_t}{\partial S_{0,t}} = N\left(d_1^{0,t}\right)$. $N\left(d_1^{1,t}\right)$ is a monotonically increasing function of $d_1^{1,t}$ which itself is monotonically increasing with respect to $S_{1,t}$. Hence, the slope of the call price curve $\Delta_t = \frac{\partial c_t}{\partial S_{0,t}}$ increases with the stock price. This result is also directly obtainable as a consequence of eq. (4).

Thus, the portfolio of X shall always and invariably generate a positive return irrespective of the return pattern of the underlying. Equivalently, a portfolio consisting of a call option shall always outperform a portfolio of comprising of the number of units of the underlying stock such that the two portfolios have the same delta.

Surely, this cannot be true. So where is the catch? Before we explore more of this, let us understand the cause of the anomaly. Let us assume that at $S_t = S_{0,t}$, Δ_t takes the value $\Delta_{0,t}$. Since the portfolio of X is delta neutral, the portfolio value will remain unchanged i.e. immunized for any infinitesimally small (emphasis placed) movements of the stock price around $S_{0,t}$. Let us, now, consider a finite increase (for concreteness) in the stock price from $S_{0,t}$ to $S_{1,t}$. This finite jump can be considered as an aggregation of an infinite stream of infinitesimal moves, say $S_{\frac{i}{n},t} - S_{\frac{i-1}{n},t}$; $i = 1,2,3,....,n$. For the first of these infinitesimal moves, say $S_{\frac{1}{n},t} - S_{0,t}$, the portfolio will stay delta neutral and hence, no change in its value will occur. Now, it is well known that in the Black Scholes Model delta is S dependent (Appendix 1). In fact, delta changes in the same direction as S. The fallout is that consequent to every infinitesimal change in S, the value of delta changes and needs to be recomputed. The portfolio of X then, must be rebalanced to keep it delta neutral. Thus, at each of the values $S_{\frac{i}{n}}$, delta will acquire a new (higher, since the stock price is rising) value. If the stock content in X's portfolio is not adjusted, the stock will fall short of the delta neutral requirement. In this case, the portfolio is no longer delta neutral, and its price will change due to a further price change in S. Additionally, because units of stock are less than the delta neutral value and stock is short, the profit due to the increase in the price of the long call will more than offset the loss due to the increase in price of ($< \Delta$) units of short stock yielding an overall profit on the portfolio. Same arguments pervade the case of a fall in the stock price.

The fallout of the analysis that we have expounded so far is:

(a) For infinitesimal price increases (decreases) of the underlying stock around $S_t = S_{0,\,t}$ the option profits (losses) are almost completely wiped out by the short stock losses (profits). Consequently, for small price moves the long volatility trade does not result in any net profit or loss. The position is perfectly hedged.

(b) (a) does not hold for large & finite price changes of the underlying stock since the delta itself is a function of the price of the stock's price. It follows that with every infinitesimal change in the stock price, delta changes. Hence, if the portfolio is not rebalanced infinitely often, it will show a resultant profit corresponding to a large & finite move of the underlying stock's price.

(c) Delta is a positive monotonic function of the underlying stock's price. Hence, as S rises, delta also increases. If portfolio rebalancing is not done, the portfolio's short stock content's delta becomes deficient of the delta-neutrality requirement. The overall (net) portfolio, thus, behaves as long in S and records gains on price rise of S. Conversely, the portfolio behaves as being short in S in the event of a price fall and again records profits.

(d) Since delta is a positive monotonic function of S, larger the magnitude of the up (down) move of S, greater would be the delta-imbalance and hence, the longness (shortness) of the net portfolio, thereby resulting in greater profits in either case.

(e) The profit generated as above, on account of a movement in the underlying stock's price shall merely remain a book entry and, as such, get wiped out on an equivalent swing in the reverse direction back to the original position. In fact, the profit will gradually get neutralized to the extent the stock price backtracks its original movement. Thus, in order that this profit may be captured by the investor, intervention needs to take place. This intervention can be in various forms, as discussed in the coming section.

Mechanics of Profit Capturing

As mentioned above, the profit generated by the long volatility trade is merely a paper profit unless it is captured by some sort of market intervention. To make the exposition tractable, we assume that the stock price has just made a spontaneous jump upwards from $S_{0,\,t}$ at $t =$ t to $S^u_{1,\,t}$. The profit generated by this move is given by eq. (1) or eq. (2) with $S_{1,\,t} \equiv S^u_{1,\,t}$. The investor at this point can either (i) do nothing in anticipation of a further price rise or (ii) capture this profit by a market intervention.

If the investor does nothing and the stock price (i) continues to increase further to, say $S^u_{2,\,t}$, the profit from the strategy also rises to that given by eq. (1) or eq. (2) with $S_{1,\,t} \equiv S^u_{2,\,t}$ (ii) starts falling from $S^u_{1,\,t}$, the profit will also start declining along with the fall in stock price, until it reaches back to $S_{0,\,t}$ at which point the entire profit will be wiped off and the original portfolio value will be restored. However, if the stock price continues to decline beyond $S_{0,\,t}$ to reach a point, say $S^d_{1,\,t}$, the profits on the strategy start getting recouped and at $S^d_{1,\,t}$ would again be given by eq. (1) or eq. (2) with $S_{1,\,t} \equiv S^d_{1,\,t}$. In this way, a passive "no action" strategy using a long volatility trade could usually result in the investor ending up seeing a paper profit, then seeing it vanish, then seeing it return and so on.

The "market intervention" for the purpose of capturing the profits can take the form of (i) either a complete liquidation of the constituents of the portfolio or (ii) a delta-based rebalancing of the portfolio. The former strategy is usually associated with transaction costs and is, relatively, expensive to implement compared to the latter which simply involves purchase/sale of the incremental units of the underlying as are essential to make the portfolio delta-neutral once again subsequent to the stock move. To elaborate, let the stock move from $S_{0,\,t}$ at $t =$ t to $S^u_{1,\,t}$ whence the delta changes from $\Delta_{0,\,t}$ to $\Delta^u_{1,\,t}$. To make the portfolio delta-neutral once again, we shall need to short additional $\Delta^u_{1,\,t} - \Delta_{0,\,t}$ units of stock at $S^u_{1,\,t}$ so that the stock component of the portfolio now has a delta of $\Delta^u_{1,\,t}$ and the overall portfolio delta vanishes. The fact that buyers of stock would still be available in the market is vindicated by the support enjoyed by the current market price. If no buyers at this price were there, this price would not be sustainable. The important point, however, is that profit is still not explicitly captured. However, to understand how this delta neutral rebalancing act leads to a formal locking in of profits, we assume that the stock after moving to $S^u_{1,t}$ (and beyond, if so) returns back to its original value of $S_{0,t}$ at which point the investor undertakes a fresh rebalancing act. Since, the overall portfolio is deficient delta of $\Delta^u_{1,t} - \Delta_{0,t}$; a purchase of $\Delta^u_{1,t} - \Delta_{0,t}$ shares at $S_{0,t}$ is mandated to make the portfolio delta-neutral once again at this point. Thus, the net effect of this pair of rebalancing transactions is that $\Delta^u_{1,t} - \Delta_{0,t}$ stock was sold short at $S^u_{1,t}$ and reacquired at $S_{0,t}$ leading to a total gain of:

$$\begin{aligned}\pi' &= \left(\Delta^u_{1,t} - \Delta_{0,t}\right)\left(S^u_{1,t} - S_{0,t}\right)\\ &= \left[N\left(d_1^{1,u,t}\right) - N\left(d_1^{0,t}\right)\right]\left(S^u_{1,t} - S_{0,t}\right)\\ &= \left(S^u_{1,t} - Ke^{-r(T-t)}\right)\left[N\left(d_1^{1,u,t}\right) - N\left(d_1^{0,t}\right)\right]\\ &\quad + \left(S_{0,t} - Ke^{-r(T-t)}\right)\left[N\left(d_1^{0,t}\right) - N\left(d_1^{1,u,t}\right)\right]\end{aligned} \tag{6}$$

The right hand side is the aggregate of the profit generated during the move from $S_{0,t}$ to $S^u_{1,t}$ (first term) and that during the return move (second term). On this rebalancing at $S_{0,t}$, the profit amount $(\Delta^u_{1,t} - \Delta_{0,t})(S^u_{1,t} - S_{0,t})$ becomes actualized from a mere paper entry.

Ideally, thus, the investor would want the stock price to repeat the pattern $S_{0,t} \to S^u_{1,t} \to S_{0,t}$ perpetually whence, for each cycle, (s) he would realize a profit $(\Delta^u_{1,t} - \Delta_{0,t})(S^u_{1,t} - S_{0,t})$. This is, however, not likely to happen always. Let us see what happens when the stock price after moving back from $S^u_{1,t}$ to $S_{0,t}$, (instead of changing direction and moving back up to $S^u_{1,t}$) continues to fall to, say, $S^d_{1,t}$ whereafter it bounces back to $S_{0,t}$ and the investor rehedges at the two points $S^d_{1,t}$ and $S_{0,t}$. This pair of actions would amount to the purchase of $(\Delta_{0,t} - \Delta^d_{1,t})$ units of stock at $S^d_{1,t}$ and resale of the same number at $S_{0,t}$ leading to the locking in of profit of:

$$\pi'' = \left(\Delta_{0,t} - \Delta^d_{1,t}\right)\left(S_{0,t} - S^d_{1,t}\right) \tag{7}$$

Thus, by using this rehedging strategy, the profits of the investor accumulate gradually if the stock price continues to oscillate from $S_{0,t}$ to $S^u_{1,t}$ to $S^d_{1,t}$ and back etc. This process of continually rehedging to form new delta neutral positions is known as dynamic hedging and this trade is called the long volatility trade as mentioned earlier. It is easy to see why this is called the long volatility trade. Justification for this nomenclature lies in the fact that the more the stock price moves and the more frequently the price reverses, the more is the scope for profits. As is obvious from the above, this trade generates profits because, by rehedging, one is forced to sell in rising markets and buy in falling markets. Thus, the investor is always trading the opposite way to the general market trend. The rehedging process forces the investor to be in the position of buying low and selling high always, thereby being able to accumulate profits on every swing that is accompanied with rehedging.

The Rehedging Frequency

However, the frequency of rehedging remains an issue. In the aforesaid exposition, the investor rehedges every time the stock price moves up to $S^u_{1,t}$, comes back to $S_{0,t}$ and either moves back up to $S^u_{1,t}$ or moves down to $S^d_{1,t}$. The irony would be if starting at $S_{0,t}$, the stock price never quite reaches either of $S^u_{1,t}$ or $S^d_{1,t}$ (the rehedging points) but falls just short of them, so that the investor never rehedges and, as such, achieves no rehedging profits. In fact, as we shall explain below, the option will suffer time decay due to finite time loss between price movements and the investor will end up with losses. The bottom line is that if the red=hedging points are much too distant apart in relation to the volatility of the stock, the stock price may not reach them frequently enough resulting in few rehedgings and scanty profits.

On the other hand, if the stock price is highly volatile and the rehedging points are kept too close, the stock may swing through point $S_{0,t}$ violently each time and move well past either $S^u_{1,t}$ or $S^d_{1,t}$ or both, before returning to $S_{0,t}$. Ideally, the rehedging should be done at the extremes of the swing of the stock price. This would result in capture of maximum profits. Thus, when $S^u_{1,t}$ or $S^d_{1,t}$ or both are too close so that they are well within the swing arc of the price, rehedging at the points $S^u_{1,t}$ or $S^d_{1,t}$ or both fails to capture the optimal amount of profits. However, the problem is that the investor does not know how far the stock price is going to move. If he did, he would not be entering into this trade. Thus, the issues of frequency of rehedging and the stock levels at rehedging are purely of judgment. Frequent rehedging results in capture of more frequent but smaller profits due to small price swings whereas occasional rehedging gives more scope for the large profits but will mean that the little profits associated with smaller price swings will be missed.

Thus, determining the frequency and price levels for rehedging is a trade off decision between the costs of rehedging and the likely time decay in the value of the option. Rehedging can be done by adjusting either the option or stock size, and costs such as the bid to offer spread, taxes and commissions will dictate which side to adjust. If these market friction losses are substantial, only a significant price move will justify the transaction charges and warrant rehedging and vice versa.

Delta, Gamma & Profits

It is seen from eq. (6) & (7) that greater the change in value of delta from its initial value $\Delta_{0,\,t}$ i.e. greater the differential $\Delta^u_{1,\,t} - \Delta_{0,\,t}$ or $\Delta_{0,\,t} - \Delta^d_{1,\,t}$, higher would be the rehedging profits. In fact, it is only due to this change in delta that rehedging profits are derived by an investor in this trade. The rate of change of delta with respect to the underlying stock's price has, therefore, a direct bearing on the quantum of rehedging profits since a higher rate would require smaller movements in the stock price to generate an equivalent amount of profit. Equivalently, higher curvature (rate of change of delta) would result in higher profits for the same amount of stock price movement. This rate is referred to as "gamma" and is invariably positive for long positions (and negative for short positions) showing that delta moves in tandem with the movements of the underlying stock's price. Explicitly, we have, in the Black Scholes Model (Appendix 1):

$$\Gamma_t = \frac{\partial \Delta_t}{\partial S_t} = \frac{\partial^2 c_t}{\partial S_t^2} = \frac{1}{\sigma S_t \sqrt{T-t}} N'\left(d_1^t\right) > 0 \qquad (8)$$

For small variations in the stock price, we may take a linear approximation and write $\left(\Delta^u_{1,t} - \Delta_{0,t}\right) = \Gamma_t\left(S^u_{1,t} - S_{0,t}\right)$ vindicating our aforesaid assertion that higher gamma would lead to a higher $\Delta^u_{1,\,t}$ corresponding to a given magnitude of stock price movement and, therefore, higher rehedging profits.

Since deep in-the-money and deep out-of-the-money options have price profiles that are nearly linear, they have very little curvature and hence, are not very useful for earning rehedging profits. Near-the-money options have the maximum curvature. Usually, gamma attains its maximum value for at-the money options. Precisely, however, the maximum value of gamma occurs at (Haug, 1997; Taleb, 1997):

$$S_t\left(\Gamma_{\max}\right) = Ke^{\left(-r-\frac{3}{2}\sigma^2\right)(T-t)} \qquad (9)$$

The relationship between the price change of a delta neutral portfolio, that of the underlying stock and the gamma and theta of that portfolio can be shown to be (Appendix 1):

$$r\Pi = \Theta + \frac{1}{2}\sigma^2 S^2 \Gamma \qquad (10)$$

and

$$\Delta\Pi = \Theta\Delta t + \frac{1}{2}\Gamma\left(\Delta S\right)^2 \qquad (11)$$

It follows from eqs. (10) & (11) that when gamma is positive, theta tends to be negative. Furthermore, the value of the delta neutral portfolio is the outcome of two opposing influences viz. the time decay or the theta effect which would push the price down for a theta negative portfolio (and vice versa) and the gamma effect which would push the price in the same direction as the gamma of the portfolio i.e. it would push the price up for positive gamma portfolios irrespective of the direction of price change of the underlying stock.

Nevertheless, the gamma values of out of the money long dated options also show a rapid increase with increased maturity. We have (Haug, 1997; Taleb, 1997)

$$\frac{\partial \Gamma}{\partial t} = \frac{N'\left(d_1\right)}{S\sigma\sqrt{T-t}}\left[\frac{rd_1}{\sigma\sqrt{T-t}} + \frac{1-d_1 d_2}{2\left(T-t\right)}\right]$$
$$= \Gamma\left[\frac{rd_1}{\sigma\sqrt{T-t}} + \frac{1-d_1 d_2}{2\left(T-t\right)}\right] \lesseqgtr 0 \qquad (12)$$

Precisely gamma has a saddle point given by:

$$\Gamma_S = \Gamma\left(S_{\Gamma_S}, T_S\right) = \frac{\sqrt{\frac{e}{\pi}}\sqrt{\frac{r}{\sigma^2}+1}}{K} \qquad (13)$$

at a time of expiry of $T_S = T - \tau_s = T - \dfrac{1}{2\left(\sigma^2 + r\right)}$

and asset price of $S_{\Gamma_\tau} = Ke^{\left(-r-\frac{3}{2}a^2\right)\tau_S}$.

The implication of this saddle point is that the maximum gamma for a given strike price is first decreasing until the saddle point, then increasing again as time to maturity decreases and the option approaches expiry. Thus, gamma is not necessarily increasing with decreasing time to maturity. The saddle point lies between the two gamma "peaks". Some market players fail to account for this issue in their trading strategies. The important point is that an illusionary feeling of risk enhancement is created by the rapid increase in gamma when long dated highly out-of the money options are added to the portfolio. Although the gamma value and hence, the gamma calibrated risk shows a rise, the actual risk, obviously, does not increase because the options added are out-of the money ones. What actually happens is that gamma is defined as the change in delta for a one unit change in the asset price. If the underlying asset's price is close to zero, a one unit movement will reflect a large change in percent of the asset price. Besides, it is also unlikely that the asset price will increase by one dollar in an instant. The definition of gamma seems to be flawed as a measure of risk to this extent and the gamma based risk measurement should be reformulated (Bates, 1991).

There could be situations when anomalous behavior occurs as a consequence of the complex relationship between the moneyness of an option; it's time to maturity and its gamma. Consider the case of a call option already out of the money. Normally, If the stock price registers further fall, the option would go more out-of-the-money and, as such, the gamma would be expected to get lower since the usual perception is that gamma was largest approximately at-the-money. However, there may be situations, depending on the time to maturity of the option, that the further out of the money the call goes the higher would be its gamma. It is common for large investment firms to monitor risk levels in terms of gamma values of the contextual portfolios. However, as seen above, adding a long maturity call option that is deep-out-of-the money (in-the-money) may enhance the gamma of the portfolio in certain circumstances, even if, in actual fact, one is close to zero gamma risk. Such high gamma risk for long dated deep-out-of-the-money options typically is only an illusion. This discrepancy in risk measurement and monitoring can be eliminated by using percentage changes in the underlying asset gammas rather than absolute values.

The "Catch" in the Long Volatility Trade: Time Decay

We, now, look at where the aforesaid analysis is flawed, for flawed it must be, because, it cannot be true to have a derivative investment strategy that invariably yields a profit, irrespective of the price process of the underlying asset. The catch is that, it was assumed throughout in the above analysis, that, as the stock price moved backward and forward, time stood still i.e. there was no lapse of time and the option did not get any older during the course of the price movements of the stock. In other words, all the stock price movements were assumed to be spontaneous. This implied that the option price profile remained fixed for ever. The cardinal implication of the stationarity of time (assumed) was that every time the stock price passed through a given point, the Black Scholes valuation of the option price was always the same. This cannot be in reality.

To explore this further, we note that, in the Black Scholes Model (Appendix 1):

$$\Theta = \frac{\partial c_t}{\partial t} = -rKe^{-r(T-t)}N\left(d_2^t\right) - S_t N'\left(d_1^t\right)\frac{\sigma}{2\sqrt{T-t}} \tag{14}$$

All the terms on the right hand side of eq. (14) must necessarily be negative whence every option loses value as it approaches maturity i.e. it undergoes "time decay" or "theta decay". Thus, in the above analysis, we need to account for the loss in option value due to the time elapsed as the share price moves between various rehedging points i.e. $S_{0,t}$, $S^u_{1,t}$ or $S^d_{1,t}$ or both etc. S_0, S^u_1, S^d_1 cannot be indexed by the same time t. This is because, in reality it would take time for the stock price to move from one price to another; In this process, other things being equal, the option would have lost some time value. And that is the catch. Every day that passes will cause the entire option price profile to decay slightly towards the kinked expiry price profile. Eq. (6), thus, gets modified. Let us assume that the stock price increases from a given S_0 at $t = t_0$ to S^u_1 at $t = t^u_1$, stays as such up to $t = t^1_1$ and then returns downwards, reaching back S_0 at $t = t^1_0$. With rebalancing being as earlier, the change in value of the portfolio will, on taking account of the time decay of the option, be:

$$\pi\left(S_0,S_1^u,t_0,t_1^u,t_1^0,t_0^1\right)=\begin{Bmatrix}\left[c\left(S_1^u,t_1^u\right)-\Delta\left(S_0,t_0\right)S_1^u\right]-\\ \left[c\left(S_0,t_0\right)-\Delta\left(S_0,t_0\right)S_0\right]\end{Bmatrix}$$

$$+\left[c\left(S_1^u,t_1^0\right)-c\left(S_1^u,t_1^u\right)\right]+\begin{Bmatrix}\left[c\left(S_0,t_0^1\right)-\Delta\left(S_1^u,t_1^u\right)S_0\right]-\\ \left[c\left(S_1^u,t_1^0\right)-\Delta\left(S_1^u,t_1^u\right)S_1^u\right]\end{Bmatrix} \tag{15}$$

so that

$$\pi\left(S_0,S_1^u,t_0,t_1^u,t_0^1\right)=\left\{\left[c\left(S_0,t_0^1\right)-c\left(S_0,t_0\right)\right]\cdot\right.$$

$$\left.+\left[\Delta\left(S_1^u,t_1^u\right)-\Delta\left(S_0,t_0\right)\right]\left(S_1^u-S_0\right)\right\} \tag{16}$$

This is the same as eq. (6) except for the first square bracketed term on the right. Now, in view of the negative nature of Θ and that $t_0 < t_0^1$, this additional term shall be negative implying that the time adjusted profits from rehedging shall necessarily be less than those without any such adjustment.

It is well known that the Black Scholes call price plot against the instantaneous underlying stock price has curvature. As the option approaches maturity, the curvature of this profile increases and, near the maturity, the curve shifts rapidly towards the kinked boundary that represents the maturity payoff indicating swift time decay. The difference between the option profiles at different time points is a measure of the time decay corresponding to a particular underlying price. Options show the maximum time decay at stock prices close to the exercise prices. The call prices relating to the extreme underlying prices all remain bunched together at the boundary and show minimal time decay. Hence, near-the-money options suffer the most time decay while deep out-of-the-money or deep in-the-money options have virtually no time decay. Unfortunately, however, the options that have maximum curvature and hence, are likely to realize maximum profits through rehedging are the ones that are near the money and as such undergo the most time decay. It is pertinent to note that time decay of an option persists even in a completely stagnant market. Long volatility traders in such situations talk of "bleeding to death" through theta.

Another Catch: The Volatility

There is another implicit assumption in the long volatility trade viz. that the volatility remains constant (Black & Scholes, 1973; Black, 1976) throughout the investment horizon. The call price invariably is a positive function of volatility, for; we have (Hull, 2012):

$$\Upsilon = S\sqrt{T-t}N'\left(d_1\right) \tag{17}$$

as, indeed, it should be since the option derives much of its "time value" from the uncertainty about the future movements of stock prices. Should the future volatility of the underlying stock's price decrease for some reason whatsoever, the call price shall immediately reflect a fall in price. This may also eat into the potential profits of the long volatility trade. The problem with volatility is more profound, because it is not directly observable, unlike time, and one has to use the best judgment estimate of projected volatility. Nevertheless, the error in estimation will result in the model returning a flawed fair value. The sensitivity of option of prices to unanticipated changes in volatility is similar to the sensitivity to time. Near-the-money options are most sensitive and vice versa. Additionally, long dated options are more influenced by errors in estimation of volatility or unanticipated changes in volatility while short dated ones are not so significantly affected, as is obvious from eq. (17).

If, after an investor enters into a volatility trade, market's estimate of the underlying stock's volatility falls, the call price and thus, portfolio value would immediately decline resulting in a fall in the rehedged profits out of the volatility trade.

Time Decay Effects on Delta & Gamma

Consider options that are deep out of the money and so are likely to remain out of the money for the future. For such options, the slope of the price profile i.e. the change in price corresponding to an infinitesimal change in price of underlying is very small and decays to zero with the passage of time. The degree of curvature Γ also gradually decreases to zero with time decay. Thus, for such options, time decay causes the stock exposure (delta) and the rate of change of the stock exposure (gamma) to decrease. An out-of-the-money long volatility position to expiry would therefore involve gradually reducing the stock hedge, to adjust for the time decay, in addition to the usual rehedging elaborated above. The scope for rehedging or delta readjusting would decrease, as the expiry of the option approaches.

In contrast to the above, the delta for deep –in- the -money options that are likely to continue in-the-money is close to one and gradually increases with the passage of time to one in the limiting case. The option, then, becomes indistinguishable from stock. However, like out- of- the money options, the curvature (Γ) gradually decreases as time passes. Thus, an in-the-money long volatility position to expiry would involve gradually increasing the stock hedge until on expiry the longs and shorts would match one for one. However, like out –of- the money trade, as expiry approaches, the gamma decreases and so the scope for rehedging would also decrease.

Near-the-money options approaching expiry show increasing curvature (Γ) with time decay, and hence, the scope for rehedging increases. In fact, close to expiry, near- the- money options can have really large gammas as the delta can swing from zero when the option is slightly out-of-the-money to one when the option is slightly in-the-money. To illustrate, consider an option which is to expire very shortly. At any $S^d < K$, the option will be completely out –of- the money and hence, insensitive to changes in stock price. The delta will, therefore, be zero at this point. Now, let the stock price tick over to any $S^u < K$, the option immediately becomes in-the- money with its value moving in equal magnitude and direction with the underlying stock's price. The delta is, therefore, one now. It follows that if in the closing minutes of the expiry cycle of the option, the stock price moved up and down around the exercise price then the rehedging activity would be enormous. Every time the stock price increased slightly one would sell short significant units of stock to rebalance delta only to buy them back if the price fell slightly. If the stock price rose slightly, delta would become one and the trader would have to sell stock short equal to the number of calls in his portfolio. Conversely, if the price fell slightly, delta would collapse to zero, and the stock requirement for hedging would be zero so all the stock sold short (at the higher price) would be re-bought (at the lower price).

Volatility Effects on Delta & Gamma

Changes in volatility have a similar effect to that of time decay on various types of options viz. options that are far out-of-the money ones have decreasing deltas and decreasing gammas with decreasing volatility whereas deep in-the-money options will have increasing deltas and decreasing gammas and near the money options show increasing gammas. In essence, a fall in volatility causes a gradually reducing option price level with more pronounced falls near-the-money. The implications are again two fold and opposing (Webb, 1999):

(a) The portfolio shall invariably suffer an immediate mark to market loss due to the option price fall.

(b) Nevertheless, if the option involved is near-the-money, this stock price fall would be accompanied by a rise in gamma, implying that there would be much more scope for rehedging than before. Although there may be an initial loss this can be more than offset by the increased chance of future rehedging.

Precisely, the sensitivity of gamma to changes in implied volatility is given by (Haug, 1997):

$$\frac{\partial\Gamma}{\partial\sigma} = \Gamma\left(\frac{d_1 d_2 - 1}{\sigma}\right) \lessgtr 0 \qquad (18)$$

The above expression is negative for prices of underlying between (S_1, S_2) and positive outside this interval, where $S_1 = Ke^{-r(T-t)-\sigma\sqrt{T-t}\sqrt{4+(T-t)\sigma^2}/2}$ and $S_2 = Ke^{-r(T-t)+\sigma\sqrt{T-t}\sqrt{4+(T-t)\sigma^2}/2}$. However, it needs to be emphasized here that these two interval limits depend on and vary with other input parameters/variables and so are merely approximate for practical use.

Volatility players would generally prefer the curvature (gamma) to increase with increase in volatility i.e. a flat volatility smile. For deep out-of-the money options, gamma increases with volatility, outside the (S_1, S_2) interval whereas it decreases for at-the money options and slightly in- or out-of the money options. In the event of the volatility being stochastic and uncorrelated with the underlying price, this provides a thumb rule for which strikes one should use higher/lower volatility when deciding on the volatility smile. In the case of volatility correlated with the asset price this becomes more complicated.

Conclusion

In the aforesaid exposition, an attempt has been made to explain the intricacies and nuances associated with the, apparently straightforward, long volatility trade. Issues of time decay and volatility estimation have been explicitly presented and their influence on the trade analyzed. Effects of delta and gamma on the strategy are also espoused.

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Appendix 1

We present below some analytical expressions for the option Greeks in the Black Scholes framework. In this model, the call price is given by (using standard notation e.g. Hull, 2012):

$$c_t = S_t N\left(d_1^t\right) - Ke^{-r(T-t)}N\left(d_2^t\right)$$

so that

$$\Delta_t = \frac{\partial c_t}{\partial S_t} = N\left(d_1^t\right) + S_t N'\left(d_1^t\right)\frac{\partial d_1^t}{\partial S_t} - Ke^{-r(T-t)}N'\left(d_2^t\right)\frac{\partial d_2^t}{\partial S_t}$$

$$= N\left(d_1^t\right) + \left[S_t N'\left(d_1^t\right) - Ke^{-r(T-t)}N'\left(d_2^t\right)\right]\frac{\partial d_1^t}{\partial S_t} \quad \text{because}$$

$$d_2^t = d_1^t - \sigma\sqrt{T-t} \text{ so that } \frac{\partial d_1^t}{\partial S_t} = \frac{\partial d_2^t}{\partial S_t}$$

$$= N\left(d_1^t\right) + \frac{1}{\sqrt{2\pi}}S_t e^{-\frac{1}{2}d_1^{t,2}}\left[1 - \frac{K}{S_t}e^{-r(T-t)+\frac{1}{2}\left(d_1^{t,2} - d_{12}^{t,2}\right)}\right]\frac{\partial d_1^t}{\partial S_t}$$

$$= N\left(d_1^t\right) + \frac{1}{\sqrt{2\pi}}S_t e^{-\frac{1}{2}d_1^{t,2}}\left[1 - \frac{K}{S_t}e^{-r(T-t)+\frac{1}{2}\left(2\sigma\sqrt{T-t}d_1^{t,} - \sigma^2(T-t)\right)}\right]\frac{\partial d_1^t}{\partial S_t}$$

$$= N\left(d_1^t\right) + \frac{1}{\sqrt{2\pi}}S_t e^{-\frac{1}{2}d_1^{t,2}}\left[1 - \frac{K}{S_t}e^{-r(T-t)-\frac{1}{2}-\sigma^2(T-t)+\ln\left(\frac{S_t}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}\right]\frac{\partial d_1^t}{\partial S_t}$$

$$= N\left(d_1^t\right) + \frac{1}{\sqrt{2\pi}}S_t e^{-\frac{1}{2}d_1^{t,2}}\left[1 - \frac{K}{S_t}e^{\ln\left(\frac{S_t}{K}\right)}\right]\frac{\partial d_1^t}{\partial S_t} = N\left(d_1^t\right). \tag{A.1}$$

$$\Gamma_t = \frac{\partial \Delta_t}{\partial S_t} = N'\left(d_1^t\right)\frac{\partial d_1^t}{\partial S_t} = \frac{1}{S_t\sigma\sqrt{T-t}}N'\left(d_1^t\right) \tag{A.2}$$

since $d_1^t = \dfrac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$ so that

$$\frac{\partial d_1^t}{\partial S_t} = \frac{1}{\sigma S_t\sqrt{T-t}}.$$

$$\Theta_t = \frac{\partial c_t}{\partial t} = S_t N'\left(d_1^t\right)\frac{\partial d_1^t}{\partial t} - Ke^{-r(T-t)}N'\left(d_2^t\right)\frac{\partial d_2^t}{\partial t}$$

$$-Ke^{-r(T-t)}N'\left(d_2^t\right)\frac{\partial d_2^t}{\partial t}$$

$$= -rKe^{-r(T-t)}N\left(d_2^t\right) + S_t N'\left(d_1^t\right)\left(\frac{\partial d_1^t}{\partial t} - \frac{\partial d_2^t}{\partial t}\right) \text{ since}$$

$$S_t N'\left(d_1^t\right) = Ke^{-r(T-t)}N'\left(d_2^t\right)$$

$$= -rKe^{-r(T-t)}N\left(d_2^t\right) + S_t N'\left(d_1^t\right)\frac{\partial}{\partial t}\left(\sigma\sqrt{T-t}\right)$$

$$= -rKe^{-r(T-t)}N\left(d_2^t\right) - S_t\frac{\sigma}{2\sqrt{T-t}}N'\left(d_1^t\right) = -Ke^{-r(T-t)}$$

$$\left[rN\left(d_2^t\right) + \frac{\sigma}{2\sqrt{T-t}}N'\left(d_2^t\right)\right] \tag{A.3}$$

$$\upsilon_t = \frac{\partial c_t}{\partial \sigma} = S_t N'\left(d_1^t\right)\frac{\partial d_1^t}{\partial \sigma} - Ke^{-r(T-t)}N'\left(d_2^t\right)\frac{\partial d_2^t}{\partial \sigma}$$

$$= S_t N'\left(d_1^t\right)\left(\frac{\partial d_1^t}{\partial \sigma} - \frac{\partial d_2^t}{\partial \sigma}\right) \text{ since}$$

$$S_t N'\left(d_1^t\right) = Ke^{-r(T-t)}N'\left(d_2^t\right)$$

$$= S_t N'\left(d_1^t\right)\frac{\partial}{\partial \sigma}\left(\sigma\sqrt{T-t}\right)$$

$$= S_t\sqrt{T-t}N'\left(d_1^t\right) = Ke^{-r(T-t)}\sqrt{T-t}N'\left(d_2^t\right) \tag{A.4}$$

$$\rho_t = \frac{\partial c_t}{\partial r} = S_t N'\left(d_1^t\right)\frac{\partial d_1^t}{\partial r} - Ke^{-r(T-t)}N'\left(d_2^t\right)\frac{\partial d_2^t}{\partial r}$$

$$+(T-t)Ke^{-r(T-t)}N\left(d_2^t\right)$$

$$= (T-t)Ke^{-r(T-t)}N\left(d_2^t\right) + S_t N'\left(d_1^t\right)\left(\frac{\partial d_1^t}{\partial r} - \frac{\partial d_2^t}{\partial r}\right)$$

since $S_t N'\left(d_1^t\right) = Ke^{-r(T-t)}N'\left(d_2^t\right)$

$$= (T-t)Ke^{-r(T-t)}N\left(d_2^t\right) + S_t N'\left(d_1^t\right)\frac{\partial}{\partial r}\left(\sigma\sqrt{T-t}\right)$$

$$= (T-t)Ke^{-r(T-t)}N\left(d_2^t\right) \tag{A.5}$$

The Black Scholes PDE is given by:

$$\frac{\partial \Pi}{\partial t} + rS\frac{\partial \Pi}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 \Pi}{\partial S^2} = r\Pi$$

In terms of option Greeks, this eq. takes the form:

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

whence, for a delta neutral portfolio, we have:

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi \qquad \text{(A.6)}$$

Considering the value of a portfolio Π as a function of the price of the underlying stock, S and time t, we can express an infinitesimal change in the value of the portfolio as a Taylor series as:

$$\Delta \prod = \frac{\partial \Pi}{\partial S}\Delta S + \frac{\partial \Pi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\Delta S^2 + \frac{1}{2}\frac{\partial^2 \Pi}{\partial t^2}\Delta t^2$$

$$+ \frac{1}{2}\frac{\partial^2 \Pi}{\partial S \partial t}\Delta S \Delta t + \ldots\ldots$$

Retaining only terms of the first order in Δt and expressing the partial derivatives in terms of the corresponding option Greeks, we have, for a delta neutral portfolio:

$$\Delta \prod = \Theta \Delta t + \frac{1}{2}\Gamma \Delta S^2 \qquad \text{(A.7)}$$