Article can be accessed online at http://www.publishingindia.com

An Analytical Approximation for Option Price under the Affine GARCH Model – A Comparison with the Closed-Form Solution of Heston-Nandi

Noureddine Lahouel*, Slaheddine Hellara*

Abstract

In the option pricing theory, two important approaches have been developed to evaluate the prices of a European option. The first approach develops an almost closed-form option pricing formula under a specific GARCH process (Heston & Nandi, 2000). The second approach develops an analytical approximation for computing European option prices with more widespread NGARCH models (Duan, Gauthier & Simonato, 1999). The analytical approximation was also developed under GJR-GARCH and EGARCH models by Duan, Gauthier, Sasseville & Simonato (2006). However, no empirical work was performed to study the comparative performance of these two formulas (closed-form solution and analytical approximation). Also, it is possible to develop an analytical approximation under the specific GARCH model of Heston & Nandi (2000). In this paper, we have filled up those gaps. We started with the development of an analytical approximation, for computing European option prices, under Heston-Nandi's GARCH model. In the second step, we carried out a comparative analysis of the three formulas using CAC 40 index returns from 31 December 1987 to 31 December 2013.

Keywords: GARCH, Option, Pricing, Approximation, Performance, Hedging

JEL Classification: C22, C32, G12, G13.

Introduction

The family of the GARCH option pricing models has occupied an important place in the empirical finance. The success of GARCH models in option valuation is due to the fact that the option valuation theory is flexible, as it can be adapted to any GARCH specification and also the GARCH processes are linked up with stochastic volatility models.

The calculation of option value in the GARCH framework is carried out by using numerical methods, which need an important time for application[1], necessitating an alternative approach to determine the option value more rapidly. Hanke (1997) proposed an approximation of the GARCH option valuation model by neural networks. A concurrent and important approach, proposed by Heston & Nandi (HN hereafter) (2000)[2], consists in developing a closed-form solution for the European options under GARCH. This method is based on the characteristic function of cumulative returns. Duan, Gauthier & Simonato (DGS hereafter) (1999)[3] proved that the closed-form GARCH option pricing formula proposed by HN (2000) is limited by one's ability to first solve the characteristic function of cumulative return analytically. This work is not possible for any of the more commonly used GARCH specifications[4]. DGS (1999) developed an analytical approximation to price European call options under the more conventional NGARCH dynamic of

---

[1] Among the numerical studies in existence, we quote Duan & Simonato (1999) and Ritchken & Trevor (1999).

[2] The Heston-Nandi's affine GARCH process has been used by several authors, as Christoffersen, Jacobs, Ornthanalai & Wang (2008), Christoffersen, Dorion, Jacobs & Wang (2010), and Christoffersen, Jacobs & Ornthanalai (2013), among others.

[3] This paper has been extended by Duan, Gauthier, Sasseville & Simonato (2006) to price European options under two other popular GARCH models, the GJR-GARCH of Glosten, Jagannathan & Runkle (1993) and the EGARCH of Nelson (1991).

[4] The dynamic of the conditional variance used by HN (2000) is engineered to yield a closed-form solution for option pricing, whereas a closed-form solution cannot be obtained for other conventional GARCH models.

---

* BESTMOD, Higher Institute of Management, Tunisa - Tunis. Email: lahouelnoureddine@gmail.com

Engle & Ng (1993). They used an Edgeworth expansion of the risk-neutral density function of returns to obtain an approximation to the European call option price. DGS used a similar approach to that of Jarrow & Rudd (1982). The approximate option pricing formula is composed of a term similar to the Black-Scholes model with the adjustment terms for skewness and kurtosis of the standardized cumulative returns. This approximation is more accurate for the short maturity options, and for the long maturity options under certain conditions.

Up to now, two main interesting questions require to be treated. There is no work performed to compare the empirical performance of the analytical approximation and the closed-form formula. An analytical approximation for European option valuation can be developed under the affine GARCH model of HN (2000). It will also be compared to the previous formulas.

The rest of this paper is organized as follows: in the second section, we present the affine GARCH model proposed by HN (2000), and the closed-form formula for call option pricing. In the section three, we develop an analytical approximation formula for European option pricing under AGARCH, using the same approach as in DGS (1999). The section four studies the numerical performance of the proposed approximation formula. The section five presents the empirical analysis of the asset returns. The comparative empirical performance, using data on CAC 40 index, will be discussed in section six. In this section, we compare the performance of the new analytical formula, in pricing and hedging options, with that of DGS (1999) and the closed-form formula of HN (2000). Finally, it is concluded in the last section.

The Heston-Nandi's Framework

The Dynamic of Returns

HN (2000) proposed a very convenient affine GARCH (AGARCH) model for the purpose of option valuation. The AGARCH model is specifically designed to yield a closed-form solution for a European option price. In this model the composed conditional return $R_{t+1} = \ln\left(S_{t+1} / S_t\right)$, where S_t is the underlying asset price at time t, are modeled as:

$$R_{t+1} = r + \lambda h_{t+1} + \sqrt{h_{t+1}}\, z_{t+1} \qquad (1)$$

With: r is the risk-free interest rate, λ is the constant price of risk, the shock z_{t+1} is assumed to be *iid* $N(0, 1)$ and h_{t+1} is the conditional variance of return on day $t+1$ which is known at the end of day t. Following is the process of h_{t+1} given by Heston-Nandi;

$$h_{t+1} = \omega + \beta h_t + \alpha\left(z_t - \delta\sqrt{h_t}\right)^2 \qquad (2)$$

The process of h_{t+1} captures time variation in the conditional variance of returns as in Engle (1982) and Bollerslev (1986), and the parameter δ captures the laverage effect. The so-called leverage effect was earlier studied by Engle & Ng (1993); as an important feature of equity returns. It captures the negative relationship between shocks to returns and volatility, which results in a negative skewed distribution of returns. The process of h_{t+1} is similar to the more conventional NGARCH of Engle & Ng (1993), which is used for option pricing by Duan (1995).

The risk-neutral dynamics for the GARCH process are given, in HN (2000), by:

$$R_{t+1} = r - \frac{1}{2} h_{t+1} + \sqrt{h_{t+1}}\,\eta_{t+1} \qquad (3)$$

$$h_{t+1} = \omega + \beta h_t + \alpha\left(\eta_t - \theta\sqrt{h_t}\right)^2 \qquad (4)$$

where $\theta = \delta + \lambda + 0.5$ measures the skewness of the risk-neutral distribution, and $\eta_t = z_t + \left(\lambda + 0.5\right)\sqrt{h_t}$ is a standard normal random variable under a locally risk-neutral measure.

In the risk-neutralized system, for option pricing, we need only four relevant parameters namely ω, β, α and θ. The volatility process is stationary if $p = \beta + \alpha\theta^2 < 1$ and the unconditional variance of the asset return is given by: $\bar{h} = \left(\omega + \alpha\right) / \left(1 - p\right)$.

The Closed-Form Formula for Option Pricing Under the AGARCH Model

In this section we present the option valuation model proposed by HN (2000). The model represents the first closed-form solution for options on spot assets whose variance follows a GARCH model. The volatility model necessary to determine the option's price is estimated and implemented solely on the basis of observable data. The model is operationally similar to the Black & Scholes

(1973) model and includes the stochastic volatility model of Heston (1993) as a continuous-time limit. Using daily data, the prices obtained by HN (2000) are numerically close to the ones obtained by Heston (1993). However, the model of HN (2000) can be more easily implemented and tested than the Heston stochastic volatility one.

Although the solution of HN (2000) is in closed form, coefficients for the generating function must be derived recursively, working backward from the time to maturity of the option. Once the generating function is derived, it is straightforward to obtain probabilities required for the call price. This requires numerical integration, but the integral representing the probabilities cannot be derived analytically. To obtain this only the case $p = q = 1$ is analyzed, corresponding to a GARCH (1, 1) process for the return variance in the HN (2000) model.

The Moment Generating Function

The moment generating function defined by $f(\varphi) = E\left[S_T^{\varphi}\right]$ can be written as follow:

$$f(\varphi) = S_t^{\varphi}\exp\left(A_t + B_t h_{t+1}\right) \tag{5}$$

Where:

$$A_t = A_{t+1} + \varphi r + \omega B_{t+1} - \frac{1}{2}\ln\left(1 - 2\alpha B_{t+1}\right) \tag{6}$$

$$B_t = \varphi(\lambda + \delta) - \frac{1}{2}\delta^2 + \beta B_{t+1} + \frac{(\varphi - \delta)^2}{2\left(1 - 2\alpha B_{t+1}\right)} \tag{7}$$

The terminal conditions of these coefficients are: $A_T = B_T = 0$

The Option Pricing Formula

The European call option, with strike price K and maturity T, is valuated at time t as:

$$C_t^{HN} = e^{-r\tau}E\left[\max\left(S_T - K, 0\right)\right] = S_t P_1 - K\exp(-r\tau)P_2 \tag{8}$$

Where $P_1 = \frac{1}{2} + \frac{e^{-r\tau}}{\pi S_t}\int_0^{\infty}\mathrm{Re}\left[\frac{K^{-i\varphi}f^*(i\varphi + 1)}{i\varphi}\right]d\varphi$ and

$$P_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left[\frac{K^{-i\varphi}f^*(i\varphi)}{i\varphi}\right]d\varphi.$$

P_2 is the risk-neutral probability of the asset price being greater than K at maturity and P_1 defines the delta of the call value. $f^*(i\varphi)$ is the conditional characteristic function of the asset price under the risk-neutral measure, and τ is the time remaining for expiration defined as $\tau = T - t$.

An Analytical Approximation for Option Pricing under the AGARCH Model

In this section, we suggest developing an analytical approximation formula, similar to that of DGS (1999) and DGSS (2006), when the dynamic of the underlying asset returns is governed by the AGARCH model of HN (2000). However, Jarrow & Rudd (1982) elaborated a theoretical framework to develop an analytical approximation formula to valuate options under general stochastic processes. They presented a technique to approximate an unknown probability distribution, called true probability distribution, with an alternative known probability distribution, called the approximating probability distribution. Using a similar approach, we can develop an analytical approximating formula to price a European call option under the AGARCH model.

Analytical Approximation Formula for a European Call Option Under AGARCH Process

Let $\rho_\tau = \ln\left(S_T/S_t\right)$, the cumulative return having a mean m_τ and a standard deviation σ_τ. Let $\mu_\tau = \left(\rho_\tau - m_\tau\right)/\sigma_\tau$ the standardized cumulative returns. The premium, at time t, of a European call option with strike price K and maturity T, can be approximated with the following formula:

$$C_t^{approx} = C + \kappa_3 A_3 + \left(\kappa_4 - 3\right)A_4 \tag{9}$$

where:

$$C = S_t\exp\left(\Delta\sigma_\tau\right)N(U) - K\exp\left(-r\tau\right)N(U - \sigma_\tau) \tag{10}$$

$$A_3 = \frac{1}{3!}S_t\sigma_\tau\exp\left(\Delta\sigma_\tau\right)\left[\left(2\sigma_\tau - U\right)n(U) + \sigma_\tau^2 N(U)\right] \tag{11}$$

$$A_4 = \frac{1}{4!}S_t\sigma_\tau\exp\left(\Delta\sigma_\tau\right)\left[\left(U^2 - 1 - 3\sigma_\tau(U - \sigma_\tau)\right)n(U) + \sigma_\tau^3 N(U)\right] \tag{12}$$

with $\tau = T - t$, $\Delta = \dfrac{m_\tau - r\tau + 0.5\sigma_\tau^2}{\sigma_\tau}$ et

$$U = \frac{\ln(S_t / K) + m_\tau + \sigma_\tau^2}{\sigma_\tau}.$$

$n(.)$ and $N(.)$ denote, respectively, the density and the cumulative functions of a standard normal random variable. The terms κ_3 and κ_4 are, respectively, the skewness and kurtosis of the standardized cumulative returns μ_τ under the risk-neutral measure:

$$\kappa_3 = E_t^*\left[\mu_\tau^3\right] \text{ and } \kappa_4 = E_t^*\left[\mu_\tau^4\right]$$

Where $E_t^*[.]$ is the expectation under the risk-neutral probability measure.

As shown by the equation (9), the analytical approximation formula of the European call option prices is composed by a term (C) similar to the formula of BS (1973) and two adjustment terms for the skewness and kurtosis of standardized cumulative returns. We can also note that in the Black and Scholes model: $m_\tau = r\tau - 0.5\sigma_\tau^2$. Generally, in the stochastic volatility context it was: $m_\tau > r\tau - 0.5\sigma_\tau^2$.

The application of this analytical approximation requires knowing the expressions of the first four moments of the cumulative return. For all maturity T, under the risk-neutral probability measure, and for all entire $k \in \{1,\ 2,\ 3,\ 4\}$:

$$E_t^*\left[\rho_\tau^k\right] = E_t^*\left[\left(\ln\frac{S_T}{S_t}\right)^k\right] = E_t^*\left[\left(r\tau - \frac{1}{2}\sum_{i=1}^{\tau} h_{t+i} + \sum_{i=1}^{\tau}\sqrt{h_{t+i}}\,\eta_{t+i}\right)^k\right] \tag{13}$$

These four moments are naturally specific to a given GARCH process. In Appendix C of DGS (1999), it was found to have some general moment formulas that require inputs specific to the GARCH process. In this paper, the analytical expressions of the moments under the AGARCH model of HN (2000) are presented in appendix.

Numerical Study of the Approximation Formula

To make a numerical analysis, we consider the set of parameters estimations, corresponding to the AGARCH and the NGARCH models, presented by table 2 in the subsection 5.2. Using these parameters, we obtain a long term variance (unconditional variance) equal to 0.000168 for the AGARCH process and 0.000195 for the NAGRCH one.

The Typical European Call Option Prices

To have an idea on the option price given by each approach as function of moneyness, figure 1 traces the evolution of this price for different values of the maturity:

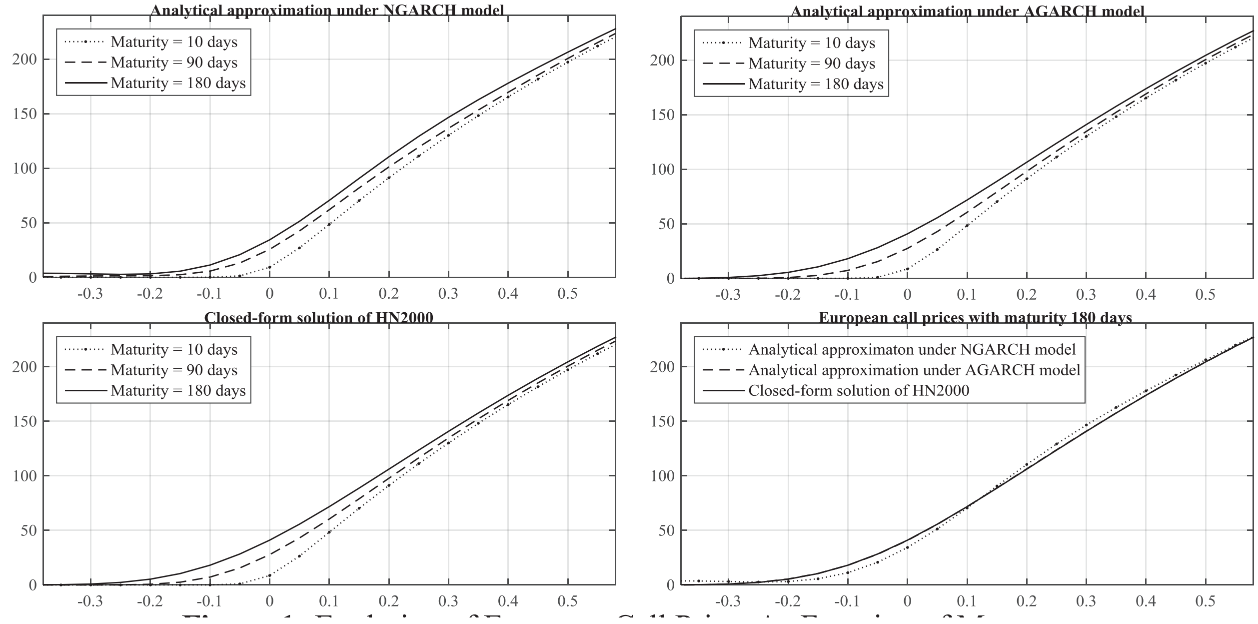

Fig. 1: **Evolution of European Call Price, As Function of Moneyness, for Different Values of Maturity**

The examination of above Figure allows us to conclude that the European call price is an increasing function of moneyness and maturity. For a maturity of six months, we compared the call prices obtained with the three approaches. Under the AGARCH model, both the new analytical approximation and the closed-form formula of (HN2000) give the almost same European call prices for all categories of moneyness. These two approaches under AGARCH model present a very small difference compared to that obtained under NGARCH model (the analytical approximation of DGS), when moneyness is less than 0.5. When the value of the moneyness is greater than 0.5, there is no difference between the three calculated prices.

Influence of the Volatility of Variance on the Call Option Prices

We studied the influence of the variance volatility on the call option price. However, the difference between the call option price given by the analytical approximation and the one given by the BS formula is represented, as a function of moneyness. The constant variance of BS is replaced by the unconditional variance from the GARCH parameters.

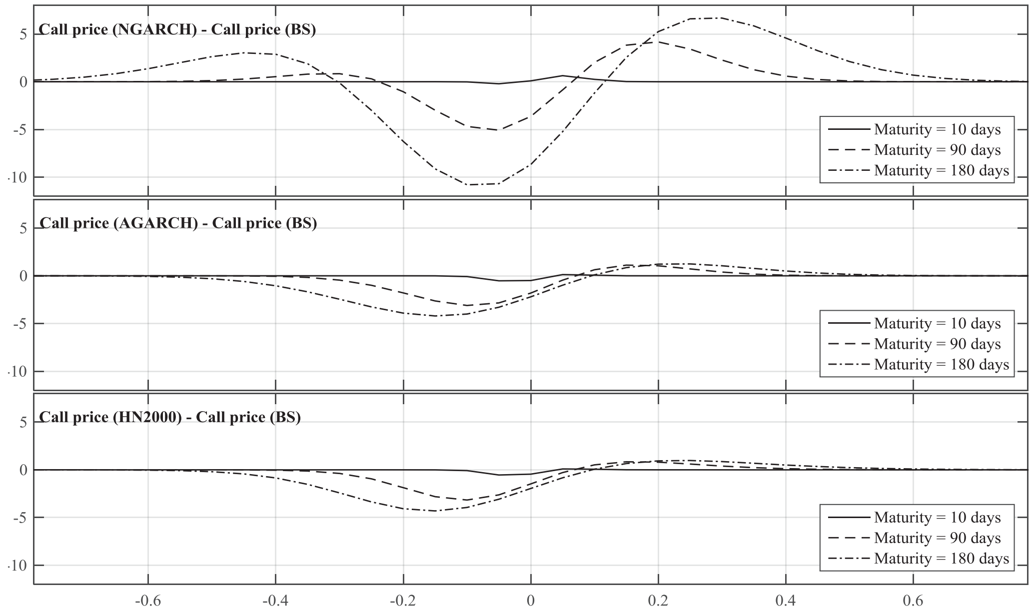

Fig. 2: **Comparison with the BS Option Price Formula**

In reality, because of the observed volatility skew, the BS model overprices ATM options and under prices deep OTM and ITM options. For that reason, the trading is more frequent for ATM options than for OTM and ITM options. However, the ATM options will have a maximum time decay[5] that leads to the highest value of the options, compared with that of OTM and ITM options.

As a preliminary remark from figure 2, we can say that the prices computed under GARCH volatility processes are more realistic than those of the BS model, for ATM and ITM options. For OTM call options, the more realistic prices are those computed by the analytical approximation under NGARCH process. When moneyness increases, there is no significant difference between call option prices given by different models. We can observe that the most important difference is obtained when the BS model is compared to the approximation formula under NGARCH model. Finally, the option price absolute difference increases with maturity for ITM options, but it decreases for OTM and ATM options.

Influence of Skewness and Kurtosis on the Option Prices

The analytical approximation formula is similar to the formula of BS, adjusted by the skewness and the kurtosis of cumulative returns. The purpose of this subsection is

---

[5] The time decay is a ratio that measures the change in option price caused by the decrease in time to maturity.

to study the influence of these shape parameters on the European call option prices. In figure 3, we represent the difference between corrected price (by skewness and kurtosis), given by the equation (9), and non-corrected price or corrected price by skewness, as function of the moneyness and maturity.

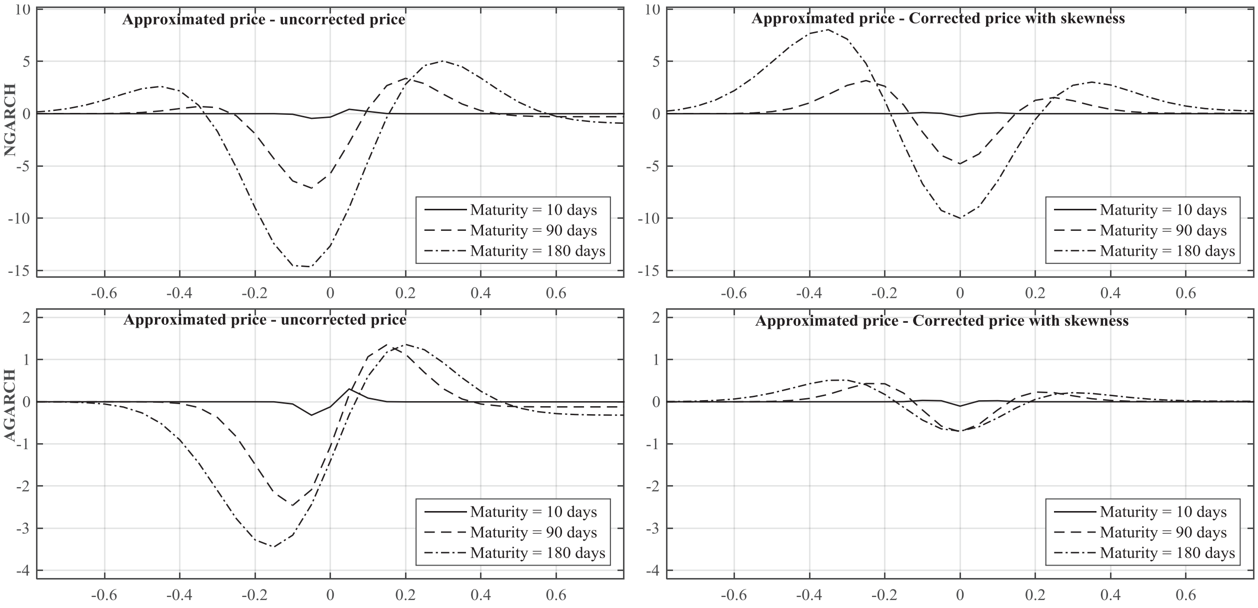

Fig. 3. **Influence of the Shape Parameters (Skewness and Kurtosis) on the Call Option Price**

When we consider the difference between approximated and non-corrected call option prices, similar conclusions to that deducted from figure 2 can be drawn. This is very logical because the uncorrected price corresponds to the term C of formula (9) which is similar to the BS formula. For the comparison between the approximated price and the corrected price by skewness, the difference is less important but has almost the same sign.

Empirical Analysis of the Asset Returns

The Data

We use data on CAC 40 index daily prices covering the period from 12/31/1987 to 12/31/2013. The risk-free interest rate is set equal to 5% per annum. Table 1 summarizes the principal descriptive statistics of CAC 40 index returns during the period under study.

Table 1. **Statistics of CAC 40 Index Returns (12/31/1987 – 12/31/2013)**

Mean	*Standard deviation*	*Skewness*	*Kurtosis*	*Q(20)*	*Q²(20)*	*JB*
0.00022	0.01392	-0.04889	7.59007	56.81346	4783.80558	5778.98582

$Q(20)$ and $Q^2(20)$ are the statistics of autocorrelation Ljung-Box test of order 20 of returns and square returns, respectively. *JB* is the Jarque-Bera statistic testing the null hypothesis considering a normal distribution of returns. The descriptive analysis results don't allow affirming that the empirical distribution of the CAC 40 index returns is assimilated to a normal distribution. This conclusion is justified by the values of skewness and kurtosis.

5.2 The Maximum Likelihood Estimation

We use the method of maximum likelihood to estimate the model parameters. Table 2 presents parameters estimates using returns data on the CAC40 for 12/31/1987 – 12/31/2013. We use a long sample of returns on the CAC40 because it permits a good estimation of GARCH models parameters, better than a short sample of data. For the risk-free interest rate r, the yearly constant rate of 5% leads to a daily rate of 5/365 = 0.0137%.

Table 2. **MLE Estimates and Properties**
(Daily returns of CAC40: 12/31/1987 – 12/31/2013)

	NGARCH(1,1)		AGARCH(1,1)	
Parameter	*Estimate*	*Robust.SE*	*Estimate*	*Robust.SE*
	1.0403e-02	5.2552e-06		2.6793e-01
λ	3.2955e-06	4.2984e-13	2.7926e-01	1.1708e-14
ω	8.6967e-01	2.1248e-04	6.7785e-11	1.8672e-04
β	6.2340e-02	9.7260e-05	8.8364e-01	3.6258e-13
α	9.0516e-01	1.1471e-02	4.4560e-06	2.8286e+02
δ			1.4205e+02	
Ln-likelihood	19759.11		19669.13	
Persistence	0.9831		0.9735	
Empirical z skewness	-0.3584		-0.3049	
Model z skewness	0.0000		0.0000	
Empirical z kurtosis	5.2497		4.9187	
Model z kurtosis	3.0000		3.0000	
Average Annual volatility	20.3121 %		19.9126 %	
Average volatility of variance	0.6917		0.4324	
Average correlation	–0.7880		–0.9171	

The results reported in Table 2 are comparable with the standard conclusions on GARCH models. The coefficients α, β and d have approximately the same importance as in the existing literature. Table 2 also reports the volatility persistence, deducted from the estimated parameters, in each model. It is important and in accordance with the literature. The NGARCH model allows us to capture larger variance persistence than its AGARCH counterpart. The robust standard errors indicate that all the estimated parameters are significant. These robust standard errors have the same importance in the two models, except for the parameters α and d. The positive estimates of l allow to guarantee positive excess log-returns. The Ln-likelihood value (obtained at the minimum of the log-likelihood function) indicates that non-affine GARCH model is preferred over the affine GARCH model. The empirical values of skewness and kurtosis of the errors z_t, compared to those of the standard normal distribution (0 and 3 respectively), denote that GARCH models can capture stylized facts of empirical returns.

Empirical Properties of Returns

To understand more the properties of the two studied models, we explore various key dynamic properties of the asset returns.

Conditional Volatility

Here we plot the annualized conditional standard deviation, as a percentage, for each model, that is, $100\sqrt{252h_{t+1}}$. The parameter values for the underlying GARCH models are obtained from maximum likelihood estimations in Table 2.

The conditional volatility patterns across the two GARCH models display similarities. However, we can see periods of high volatilities followed by periods of low volatilities, and vice versa. This phenomenon is called volatility clustering, which is an important stylized fact of asset returns. We can therefore, say that GARCH processes can adjust well with the empirical asset returns. Figure 4 reveals an important difference among the considered GARCH processes. The NGARCH appears to display, sometimes, much more variation in the conditional volatility than the AGARCH model.

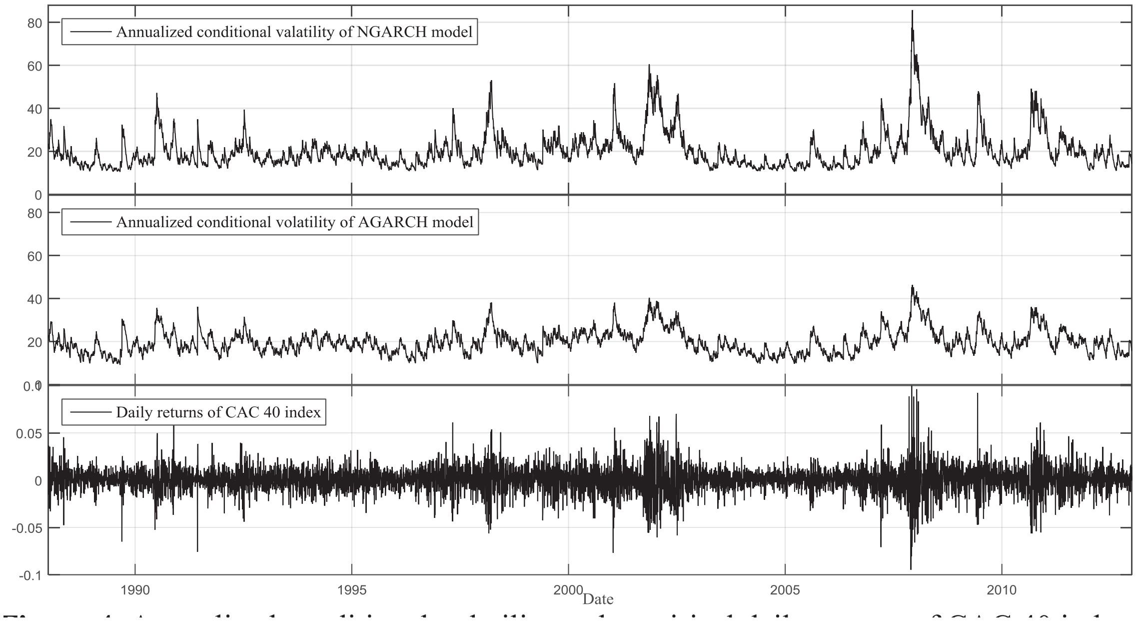

Fig. 4: Annualized Conditional Volatility and Empirical Daily Returns of CAC 40 Index

Conditional Volatility of Variance

We plot here the annualized conditional volatility of variance using the parameter values of Table 2. However, the future variance h_{t+2} is stochastic and its probability distribution is useful for option valuation. The conditional variance of h_{t+2} can be derived easily, as follows:

$$Var_t\left(h_{t+2}\right)=\alpha^2\left(2+4\delta^2 h_{t+1}\right) \qquad (14)$$

under the AGARCH model, and

$$Var_t\left(h_{t+2}\right)=\alpha^2\left(2+4\delta^2\right)h_{t+1}^2 \qquad (15)$$

with the NGARCH process.

The annualized conditional volatility of variance as a percentage, given by $100*252*\sqrt{Var_t\left(h_{t+2}\right)}$, is driven by the α parameter in the AGARCH and the NGARCH models.

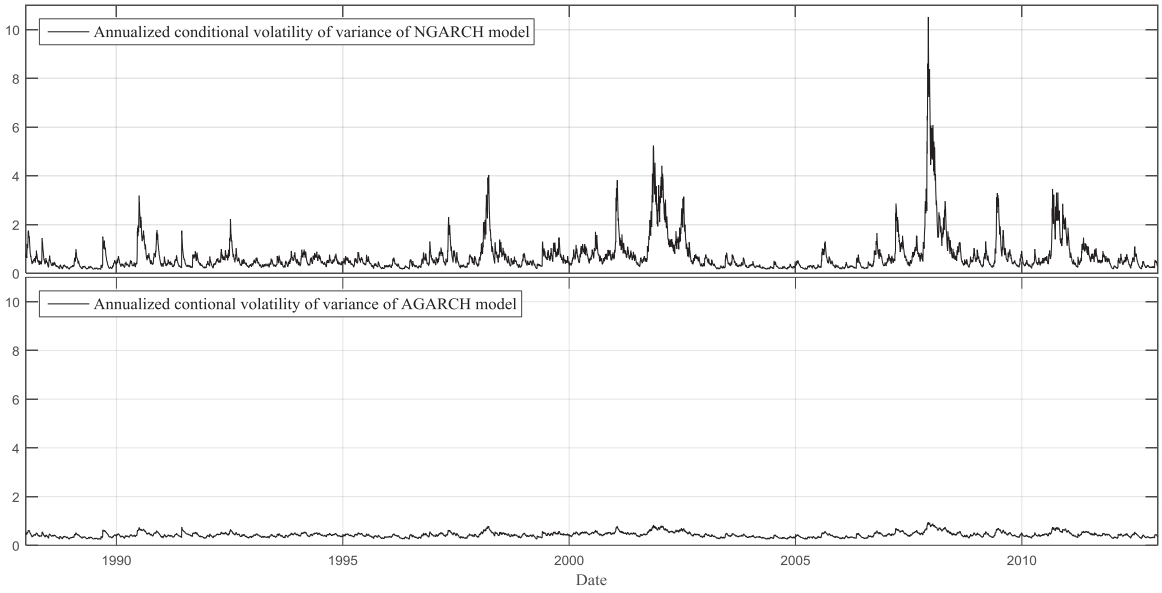

Fig. 5: Annualized Conditional Volatility of Variance

Figure 5 demonstrates that the NGARCH model has a larger volatility of variance than its AGARCH counterpart. Table 2 also reports that the average annualized conditional volatility of variance, which is greater in NGARCH framework than in that of AGARCH.

Conditional Correlation

We here plot the conditional correlation between return at time $t + 1$ and its conditional variance at time $t + 2$, which is given by:

$$Corr_t\left(R_{t+1},h_{t+2}\right)=\frac{-2\delta\sqrt{h_{t+1}}}{\sqrt{2+4\delta^2 h_{t+1}}} \tag{16}$$

under AGARCH model and

$$Corr_t\left(R_{t+1},h_{t+2}\right)=\frac{-2\delta}{\sqrt{2\left(1+2\delta^2\right)}} \tag{17}$$

in NGARCH model.

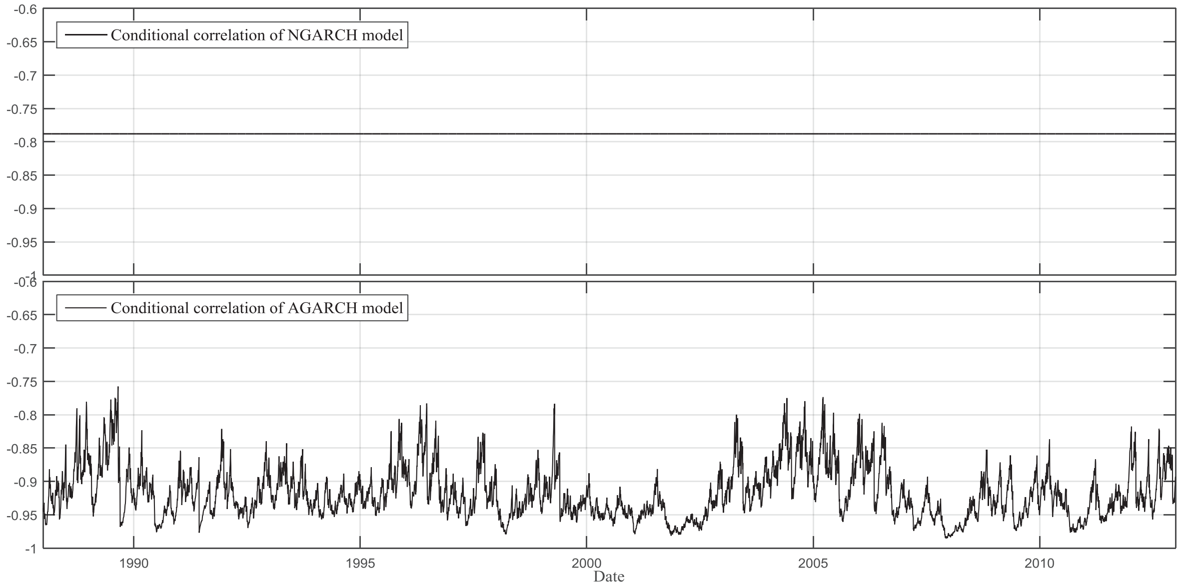

Fig. 6: **Conditional Correlation between Returns and Variance**

These conditional correlations are negative because they are driven by the parameter d, which is, significantly positive in all cases. Note that the NGARCH model implies a constant conditional correlation. The AGARCH model instead has time-varying conditional correlation, sometimes very close to -1. From table 2, we can say that the average conditional correlation between return and conditional variance is more negative in AGARCH model than in NGARCH model. This result demonstrates that AGARCH model displays a more pronounced leverage effect than NGARCH model.

The Variance Term Structure

In this subsection we show graphically an important statistical property for the option pricing. We illustrate the variance term structures (VTS) of the studied GARCH models using parameters estimations in table 2. However, the convenient measure of the variance term structure for maturity T is given by:

$$\bar{h}_t=\frac{1}{\tau}\sum_{s=1}^{\tau}E\left[h_{t+s}\right]=\bar{h}+\left(\frac{1-p^{\tau}}{1-p}\right)\left(\frac{h_{t+1}-\bar{h}}{\tau}\right) \tag{18}$$

where p denotes the first-order process of stationarity of the variance (the variance has the first-order weak stationarity if $p < 1$). It is given by $p=\beta+\alpha(\lambda+\delta+0.5)^2$ in the AGARCH model of HN (2000), and $p=\beta+\alpha\left(1+(\lambda+\delta)^2\right)$ in the NGARCH model of Engle & Ng (1993).

The variance term structure gives important information about the model's potential to explain the variation of option prices across maturity.

To compare different models, we can set the current variance; h_{t+1}, to a simple multiple of the long run variance

$\left(h_{t+1}=m\bar{h}\right)$. In this case the variance term structure relative to the unconditional variance is:

$$\bar{h}_t/\bar{h}=1+\left(\frac{1-p^{\tau}}{1-p}\right)\left(\frac{m-1}{\tau}\right) \tag{19}$$

Following Christoffersen, Jacobs, Ornthanalai & Wang (2008), we set m = 1/2to investigate a low (dashed line) initial variance and m = 2 to investigate a high (solid line) initial variance. In figure 7, we can see clearly that the conditional variance converges to the long-run variance (unconditional variance) for the two models. Also, figure 7 shows that the conditional variance converges to its unconditional level more quickly in the AGARCH model than in the NGARCH model.

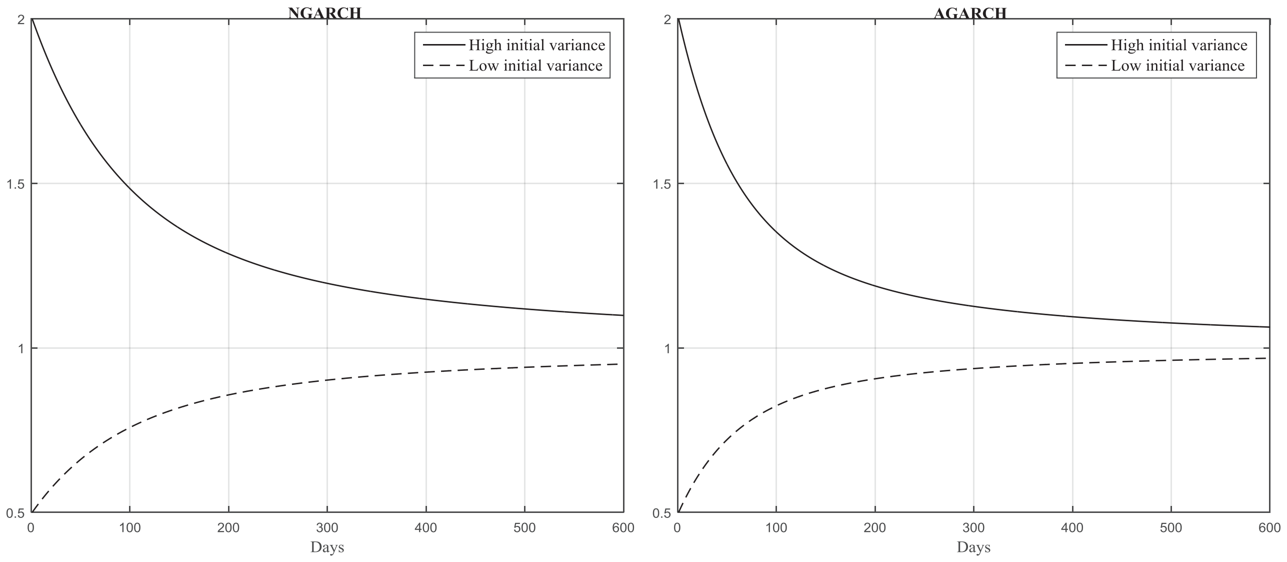

Fig. 7: **Variance Term Structure**

Empirical Performance in Pricing and Hedging Options

To price options using the approaches discussed earlier, we study the performance of each analytical approximation, developed under affine and non-affine GARCH models, compared to the closed-form formula of HN (2000). In a first step, we examine the price of European call as function of moneyness and maturity[6], using the parameter estimations from table 2. The second step studies the static performance of all approaches in option valuation. Finally, the third step presents the dynamical performance in optional portfolio hedging.

The Option Pricing Performance

In this subsection the main goal is to discuss the model performance in forecasting the option prices. However, we use parameters estimated at time t-1 to calculate the option price at time t. In fact, the instantaneous knowledge of the parameter values is difficult for the market operators. For that, one uses the parameter estimations obtained on the previous date. The used underlying asset price is that of the current date. The calculated option price will be compared to the one observed at the same time and having the same maturity. The difference, between expected option price and the correspondent observed one, gives the forecasting error of the adopted model. The obtained results are given in the following table[7]:

Table 3: **Out-of-sample Forecasting Errors**

	< 1 month		*1–2 months*		*2–6 months*	
	Average	*Std. dev*	*Average*	*Std. dev*	*Average*	*Std. dev*
NGARCH	0.6573	0.2361	0.5379	0.1403	0.7793	0.0809
AGARCH	0.5321	0.3093	0.4990	0.2324	0.9759	0.2511
HN2000	0.5484	0.3151	0.5191	0.2404	0.9925	0.2545
BS	0.6569	0.2782	0.6217	0.1894	1.1125	0.1720

---

[6] We studied options with maturity greater than 5 days, and we considered three maturity buckets, of less than 1 month, between 1 and 2 months and between 2 and 6 months.

[7] The results reported in table 3, and represented by figure 8, relate to the at-the-money options.

From table 3, we denote that the best forecasting performance is when the maturity is between one and two months. The more efficient approach is the AGARCH approximation for the first and the second maturity buckets, and the NGARCH approximation for the third maturity bucket (between 2 and 6 months). The analytical approximation under AGARCH is better than the closed-form solution of HN2000. The BS approach is dominated by all other methods. The obtained results are visualized by the following figure:

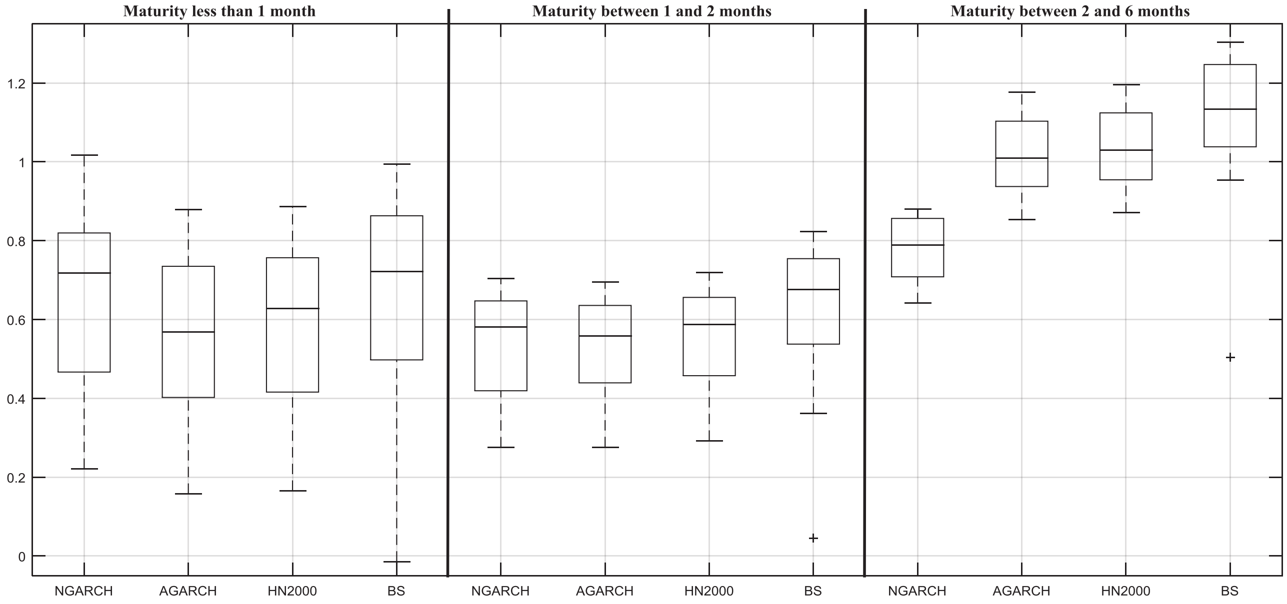

Fig. 8: **Box-plots of Option Pricing Errors**

The Option Hedging Performance

The Hedging Strategy

To evaluate the hedging performance of our new approximated formula, we follow the approach of Duan, Ritchken and Sun (2006). We consider a call option that is to be hedged over *n* successive days. At the date *t*, the discrete delta hedged results (hedging errors), over the *n* days, is given by:

$$\pi_t = \left(C_{t+n} - C_t\right) - \sum_{i=1}^{n} \Delta_{t+i-1}\left(S_{t+i} - S_{t+i-1}\right) - \sum_{i=1}^{n} r\left(C_t - \Delta_{t+i-1} S_{t+i-1}\right) \tag{20}$$

Where Δt is the delta hedge ratio given by the valuation model at time *t*. It defines the proportion of underlying asset in the replication portfolio. However, the delta of a call option with price C_t is defined as:

$$\Delta_t = \partial C_t / \partial S_t \tag{21}$$

In the BS framework, the delta hedge ratio is:

$$\Delta_t^{BS} = N(d_1) \tag{22}$$

Where $N(.)$ is the standard normal cumulative distribution and $d_1 = \dfrac{\ln(S_t / K) + (r + 0.5\bar{\sigma}_t^2)\tau}{\bar{\sigma}_t \sqrt{\tau}}$.

With $\bar{\sigma}_t$ is the implied volatility that equates the observed option price to the theoretical one at date *t*.

For our analytical approximation formula, the hedge ratio is derived as follow:

$$\Delta_t^{approx} = \frac{\partial C_t^{approx}}{\partial S_t} = \frac{\partial C}{\partial S_t} + \kappa_3 \frac{\partial A_3}{\partial S_t} + \left(\kappa_4 - 3\right)\frac{\partial A_4}{\partial S_t}$$

$$= \frac{\exp(\Delta\sigma_\tau)}{\sigma_\tau}\left[n\left(d\right) + \sigma_\tau N\left(d\right)\right] - \frac{K\exp(-r\tau)}{\sigma_\tau S_t} n\left(d - \sigma_\tau\right)$$

$$+ \kappa_3 \frac{\exp(\Delta\sigma_\tau)}{3!}\left[\left(d^2 - 3\sigma_\tau d + 3\sigma_\tau^2 - 1\right) n\left(d\right) + \sigma_\tau^3 N\left(d\right)\right]$$

(23)

$$+ \left(\kappa_4 - 3\right)\frac{\exp(\Delta\sigma_\tau)}{4!}\left[\left(4\sigma_\tau^3 - 4\sigma_\tau + 3(1 - 2\sigma_\tau^2)d +\right.\right.$$

$$+4\sigma_\tau d^2 - d^3\Big) n(d) + \sigma_\tau^4 N(d)\Big]$$

For the closed-form solution, HN2000 argued that the delta hedge parameter is given by:

$$\Delta_t^{HN} = P_1 = \frac{1}{2} + \frac{e^{-r\tau}}{\pi S_t}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\varphi} f^*(i\varphi+1)}{i\varphi}\right] d\varphi \qquad (24)$$

The evolution of the delta hedge ratio, as a function of moneyness and maturity, is represented by the following figure:

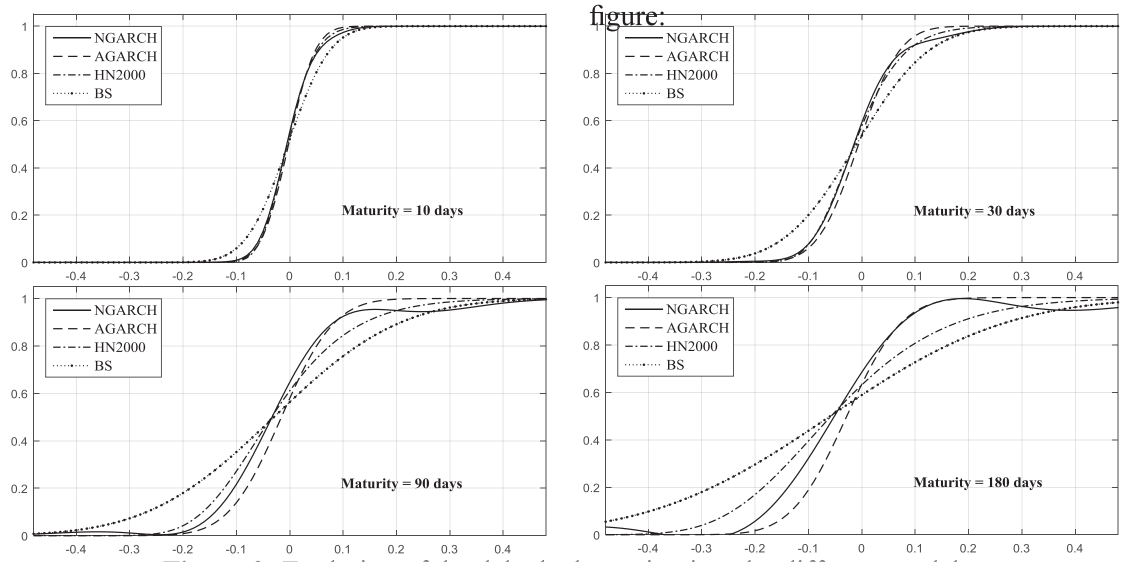

Fig. 9: **Evolution of the Delta Hedge Ratio Given by Different Models**

From figure 9, we conclude that the delta hedge ratio is an increasing (decreasing) function of maturity for OTM (ITM) options. It tends to have the same value for ATM options, whatever may be the maturity value. The highest value of delta for OTM options is that given by the BS model, which has the lowest value for ITM options. The proposed analytical approximation gives the lowest delta hedge ratio for OTM options and the highest one for ITM options.

Out-of-sample Hedging Performance

Figure 10 shows the box-plots of the hedging errors, using delta hedge values computed by different valuation approaches. The hedging results are produced for ATM European call options. The figure shows that the hedging performance, given by the delta hedging strategies, is similar for all GARCH based models. Only the BS model shows significant difference with the other models. All the GARCH based models produced hedge results better than the BS model. All models produced better hedge results when maturity increases.

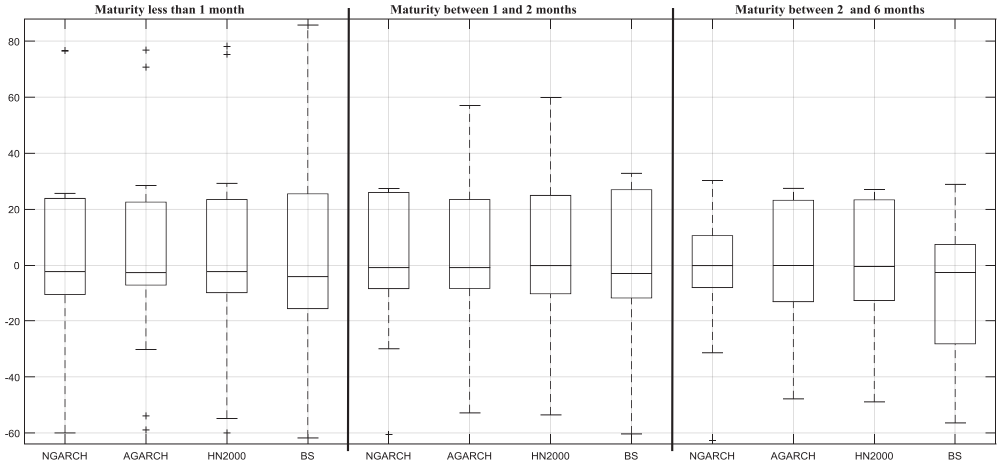

Fig. 10: **Box-plots of the Raw Hedging Errors**

Conclusion

In this paper, we developed an analytical approximation formula allowing the valuation of a European call option under the affine GARCH model of HN2000. In fact, to have this analytical option-pricing formula, we used the general theoretical framework elaborated by Jarrow & Rudd (1982) and applied by DGS1999 to NGARCH model and by Duan, Gauthier, Sasseville and Simonato (2006) to EGARCH and GJR-GARCH models. This technique allows to approximate an unknown probability distribution (true distribution) using an alternative distribution (approximating distribution). However, doing the approximation of the standardized cumulative return distribution by the normal distribution, we have managed to establish the analytical approximation of the European call option price under the AGARCH model of HN2000.

The confrontation of the proposed analytical formula under the affine GARCH model, with the analytical approximation under NGARCH and the closed-form formula has been carried out using realistic data on CAC 40 index. The results showed that our developed formula is better than the closed-form solution in option pricing.

For the portfolio hedging, using the delta hedging approach of Duan, Ritchken and Sun (2006), this study shows that there is no significant difference between the GARCH based models.

Looking forward, following works in this framework are likely to be developed. We can quote some subsequent research prospects:

- The first development concerns applying the proposed analytical approximation on other type of data.
- The second direction can concern comparing this new formula with continuous-time models like Bakshi, Cao & Chen (1997).
- The third problematic suggests the development of the analytical approximation using non-normal distribution of the random variable z_t. As an example, we can reflect to the GED distribution.

References

Bakshi, G., Cao, C., & Chen, Z. (1997). Empirical performance of alternative option pricing models. *Journal of Finance,* 52, 2003-2049.

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy,* 81, 637-654.

Bollerslev, T. (1986). Generalized auto regressive conditional heteroskedasticity. *Journal of Econometrics,* 31, 307-327.

Christoffersen, P., Dorion, C., Jacobs, K., & Wang, K., (2010). Volatility components: Affine restrictions and non-normal innovations. *Journal of Business and Economic Statistics,* 28, 483-502.

Christoffersen, P., Jacobs, K., & Ornthanalai, C., (2013). GARCH Option Valuation: Theory and Evidence. *Journal of Derivatives,* 21, 8-41.

Christoffersen, P., Jacobs, K., Ornthanalai, C., & Wang, Y. (2008). Option valuation with long-run and short-run volatility components. *Journal of Financial Economics,* 90, 272-297.

Duan, J. C. (1995). The GARCH option pricing models. *Mathematical Finance*, 5, 13-32.

Duan, J. C., Gauthier, G., & Simonato, J. G. (1999). An analytical approximation for the GARCH option pricing model. *Journal of Computational Finance*, 2(4), 75-116.

Duan, J. C., Gauthier, G., Sasseville, C., & Simonato, J. G. (2006). Approximating the JGR-GARCH and EGARCH option pricing models analytically. *Journal of Computational Finance,* 9(3).

Duan, J. C., Ritchken, P., & Sun, Z. (2007). Jump Starting GARCH: Pricing Options with Jumps in Returns and Volatilities, Working Paper.

Duan, J. C., & Simonato, J. G. (1999). American option pricing under GARCH by a Markov chain approximation, working paper, Hong Kong University of Science and Technology.

Engle, R. (1982). Auto regressive conditional heteroskedasticity with estimates of the variance of UK inflation. *Econometrica,* 50, 987-1008.

Engle, R., & Ng, V. (1993). Measuring and testing the impact of news on volatility. *Journal of Finance,* 48, 1749-1778.

Glosten, L. R., Jagannathan, R., & Runkle, D. E. (1993). On the relation between the expected value and volatility of the nominal excess return on stocks. *Journal of Finance,* 48, 1779-1801.

Hanke, M. (1997). Neutral network approximation of option pricing formulas for analytically intractable option pricing models. *Journal of Computational Intelligence in Finance,* 5, 20-27.

Heston S. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies,* 6, 327-343.

Heston, S., & Nandi, S. (2000). A closed-form GARCH option pricing model. *Review of Financial Studies,* 13, 585-626.

Jarrow, R. A., & Rudd, A. (1982). Approximate option valuation for arbitrary stochastic processes. *Journal of Financial Economics,* 10, 347-369.

Nandi, S. (1996). Pricing and hedging index options under stochastic volatility: An empirical examination. Working paper, Federal Reserve of Atlanta.

Nelson, D. (1991). Conditional heteroskedasticity in asset returns: A new approach. *Econometrica,* 59, 347-370.

Ritchken, P., & Trevor R. (1999). Pricing options under generalized GARCH and stochastic volatility processes. *Journal of Finance,* 54, 377-402.

Appendix

From equation (4) of the conditional variance, for any positive integers *s* and *t*, we can prove that:

$$h_{s+t} = \omega\left(\frac{1-\gamma^t}{1-\gamma}\right) + \gamma^t h_s + \alpha\gamma^t \sum_{r=1}^{t} \gamma^{-r} y_{s+r-1}$$

where: $\gamma = \beta + \alpha\theta^2$ and $y_t = \eta_t^2 - 2\theta h_t^{1/2}\eta_t$.

The moments of y_t are:

$$E(y_t) = 1 \quad ; \quad E\left(y_t^2\right) = 3 + 4\theta^2 E(h_t) \quad ; \quad E\left(y_t^3\right) = 15 + 36\theta^2 E(h_t)$$

As in DGS (1999) and DGSS (2006), some terms in the approximation formula have been dropped because they have negligible effects on the quality of approximation. The dropped terms are S_{Q_1} (all terms), S_{Q3} (all terms), S_{Q4} (except for terms 8 and 12) and S_{Q5} (except for terms 2, 3, 6, 7 and 8). The remaining terms are determined from the following formulas:

$$\bullet E(h_i) = \bar{h} + \left(h_1 - \bar{h}\right)\gamma^{i-1} \; ; \quad \bar{h} = \frac{\omega+\alpha}{1-\gamma}$$

$$\bullet E\left(h_i^2\right) = \left(\omega\left(\frac{1-\gamma^{i-1}}{1-\gamma}\right) + \gamma^{i-1}h_1\right)^2 + 2\alpha\left(\omega\left(\frac{1-\gamma^{i-1}}{1-\gamma}\right) + \gamma^{i-1}h_1\right)\left(\frac{1-\gamma^{i-1}}{1-\gamma}\right)$$
$$+\alpha^2\left(\left(\frac{2}{\gamma-1}\right)\left(\frac{1-\gamma^{2i-2}}{1-\gamma^2} - \frac{1-\gamma^{i-1}}{1-\gamma}\right) + \left(3+4\theta^2\bar{h}\right)\left(\frac{1-\gamma^{2i-2}}{1-\gamma^2}\right) + 4\theta^2\left(h_1-\bar{h}\right)\left(\frac{\gamma^{i-1}-\gamma^{2i-2}}{\gamma-\gamma^2}\right)\right)$$

$$\bullet E\left(h_i^3\right) = \left(\omega\left(\frac{1-\gamma^{i-1}}{1-\gamma}\right) + \gamma^{i-1}h_1\right)^3 + 3\alpha\left(\omega\left(\frac{1-\gamma^{i-1}}{1-\gamma}\right) + \gamma^{i-1}h_1\right)^2\left(\frac{1-\gamma^{i-1}}{1-\gamma}\right)$$
$$+3\alpha^2\left(\omega\left(\frac{1-\gamma^{i-1}}{1-\gamma}\right) + \gamma^{i-1}h_1\right)\left(\begin{array}{l}\left(\frac{2}{\gamma-1}\right)\left(\frac{1-\gamma^{2i-2}}{1-\gamma^2} - \frac{1-\gamma^{i-1}}{1-\gamma}\right) + \left(3+\square\theta^2\bar{h}\right)\left(\frac{1-\gamma^{2i-2}}{1-\gamma^2}\right) \\ +\square\theta^2\left(h_1-\bar{h}\right)\left(\frac{\gamma^{i-1}-\gamma^{2i-2}}{\gamma-\gamma^2}\right)\end{array}\right)$$

$$+\alpha^3\left(\begin{array}{l}\frac{6}{1-\gamma}\left(\frac{1}{1-\gamma^2}\left(\frac{1-\gamma^{3i-3}}{1-\gamma^3}-\frac{1-\gamma^{i-1}}{1-\gamma}\right)-\frac{1}{1-\gamma}\left(\frac{1-\gamma^{2i-2}}{1-\gamma^2}-\frac{1-\gamma^{i-1}}{1-\gamma}\right)\right)\\ +\frac{3}{1-\gamma}\left(\left(3+4\theta^2\bar{h}\right)\left(\frac{1-\gamma^{3i-3}}{1-\gamma^3}-\frac{1-\gamma^{2i-2}}{1-\gamma^2}\right)+4\theta^2\left(h_1-\bar{h}\right)\left(\frac{\gamma^{i-1}-\gamma^{3i-3}}{\gamma-\gamma^3}-\frac{\gamma^{i-1}-\gamma^{2i-2}}{\gamma-\gamma^2}\right)\right)\\ +3\left(\begin{array}{l}\left(\frac{3+4\theta^2\bar{h}}{\gamma^2-1}\right)\left(\frac{1-\gamma^{3i-3}}{1-\gamma^3}-\frac{1-\gamma^{i-1}}{1-\gamma}\right)\\ +\frac{4\theta^2}{\gamma^2-\gamma}\left(\left(h_1-\bar{h}\right)\left(\frac{\gamma^{i-1}-\gamma^{3i-3}}{1-\gamma^2}-\frac{\gamma^{i-1}-\gamma^{2i-2}}{1-\gamma}\right)+2\alpha\left(\frac{1-\gamma^{3i-3}}{1-\gamma^3}-\frac{1-\gamma^{2i-2}}{1-\gamma^2}\right)\right)\\ +\frac{16\alpha\theta^5}{\gamma^2-\gamma}\left(\bar{h}\left(\frac{1-\gamma^{3i-3}}{1-\gamma^3}-\frac{1-\gamma^{2i-2}}{1-\gamma^2}\right)+\left(h_1-\bar{h}\right)\left(\frac{\gamma^{i-1}-\gamma^{3i-3}}{\gamma-\gamma^3}-\frac{\gamma^{i-1}-\gamma^{2i-2}}{\gamma-\gamma^2}\right)\right)\end{array}\right)\\ +\left(15+36\theta^2\bar{h}\right)\left(\frac{1-\gamma^{3i-3}}{1-\gamma^3}\right)+36\theta^2\left(h_1-\bar{h}\right)\left(\frac{\gamma^{i-1}-\gamma^{3i-3}}{\gamma-\gamma^3}\right)\end{array}\right)$$

$$\bullet E\left(h_i^p\eta_i^q h_{i+j}\right)=\left[\bar{h}\left(1-\gamma^j\right)-\alpha\gamma^{j-1}\right]E\left(h_i^p\eta_i^q\right)+\gamma^j E\left(h_i^{p+1}\eta_i^q\right)+\alpha\gamma^{j-1}\left[E\left(h_i^p\eta_i^{q+2}\right)-2\theta E\left(h_i^{p+1/2}\eta_i^{q+1}\right)\right]$$

$$\bullet E\left(h_i^p z_i^q h_{i+j}^2\right)=\left(\begin{array}{l}\omega^2\left(\frac{1-\gamma^j}{1-\gamma}\right)^2+2\omega\alpha\left(\frac{1-\gamma^j}{1-\gamma}\right)\left(\frac{\gamma-\gamma^j}{\gamma-\gamma^2}\right)+\frac{2\alpha^2}{\gamma-1}\left(\frac{\gamma^2-\gamma^{2j}}{\gamma^2-\gamma^4}-\frac{\gamma-\gamma^j}{\gamma-\gamma^2}\right)+3\alpha^2\left(\frac{\gamma^2-\gamma^{2j}}{\gamma^2-\gamma^4}\right)\\ +\frac{4\alpha^2\theta^2}{1-\gamma}\left(\omega\left(\frac{\gamma^2-\gamma^{2j}}{\gamma^2-\gamma^4}-\frac{\gamma^{j+1}-\gamma^{2j}}{\gamma^2-\gamma^3}\right)+\alpha\left(\frac{\gamma^2-\gamma^{2j}}{\gamma^2-\gamma^4}-\frac{\gamma^{j+1}-\gamma^{2j}}{\gamma^3-\gamma^4}\right)\right)\end{array}\right)E\left(h_i^p\eta_i^q\right)$$

$$+\left(2\omega\left(\frac{\gamma^j-\gamma^{2j}}{1-\gamma}\right)+2\alpha\left(\frac{\gamma^{j+1}-\gamma^{2j}}{\gamma-\gamma^2}\right)+4\alpha^2\theta^2\left(\frac{\gamma^{j+1}-\gamma^{2j}}{\gamma^2-\gamma^3}\right)\right)E\left(h_i^{p+1}\eta_i^q\right)+\gamma^{2j}E\left(h_i^{p+2}\eta_i^q\right)$$

$$+\left(2\omega\alpha\left(\frac{\gamma^j-\gamma^{2j}}{\gamma-\gamma^2}\right)+2\alpha^2\left(\frac{\gamma^{j+1}-\gamma^{2j}}{\gamma^2-\gamma^3}\right)+4\alpha^3\theta^2\left(\frac{\gamma^{j+1}-\gamma^{2j}}{\gamma^3-\gamma^4}\right)\right)E\left(h_i^p\eta_i^{q+2}\right)$$

$$+\left(2\alpha\gamma^{2j-1}+4\alpha^2\theta^2\gamma^{2j-2}\right)E\left(h_i^{p+1}\eta_i^{q+2}\right)+\alpha^2\gamma^{2j-2}E\left(h_i^p\eta_i^{q+4}\right)$$

$$-4\alpha\theta\left(\omega\left(\frac{\gamma^j-\gamma^{2j}}{\gamma-\gamma^2}\right)+\alpha\left(\frac{\gamma^{j+1}-\gamma^{2j}}{\gamma^2-\gamma^3}\right)+2\alpha^2\theta^2\left(\frac{\gamma^{j+1}-\gamma^{2j}}{\gamma^3-\gamma^4}\right)\right)E\left(h_i^{p+1\square 2}\eta_i^{q+1}\right)$$

$$-4\alpha\theta\gamma^{2j-1}E\left(h_i^{p+3\square 2}\eta_i^{q+1}\right)-4\alpha^2\theta\gamma^{2j-2}E\left(h_i^{p+1\square 2}\eta_i^{q+3}\right)$$

$$\bullet E\left(h_i^p\eta_i^q h_{i+j}^r\eta_{i+j}^m h_{i+j+k}\right)=\left[\bar{h}\left(1-\gamma^k\right)-\alpha\gamma^{k-1}\right]E\left(h_i^p\eta_i^q h_{i+j}^r\eta_{i+j}^m\right)+\gamma^k E\left(h_i^p\eta_i^q h_{i+j}^{r+1}\eta_{i+j}^m\right)$$

$$+\alpha\gamma^{k-1}\left[E\left(h_i^p\eta_i^q h_{i+j}^r\eta_{i+j}^{m+2}\right)-2\theta E\left(h_i^p\eta_i^q h_{i+j}^{r+1/2}\eta_{i+j}^{m+1}\right)\right]$$