Journal of Supply Chain Management System
7 (3) 2018, 16-22
http://publishingindia.com/jscms/

Production Markovian Inventory Model With Baye's Estimation

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Abstract

A Markovian inventory periodic review production model with inter-demand time as exponential distribution is considered in this paper. The model is developed on the basis of constant production rate and constant rate of deteriorating items without shortages. The unit production cost is inversely proportional to the demand rate. The model contains the exponential parameter which is unknown and is estimated through MLE and Baye's under a squared error loss function. The conjugate Gamma prior is used as the prior distribution of exponential distribution. Finally, a numerical MCMC simulation is used to compare the estimators obtained with Expected risk and are shown graphically. The objective of the paper is to develop an optimum policy that minimizes the total average cost by using the above estimates of the parameter. The sensitivity analysis is also carried out for the model with percentage change in the parameters.

Keywords: Baye's Estimation, Deteriorating Items, Exponential Distribution, Optimal Time Periods, Squared Loss Function, Stochastic Demand

Introduction

Inventory system is one of the main streams of the operations research which is essential in business enterprises and industries. More operations research has been directed towards inventory control than toward any other problem area in business and industry. It is usually large quantity of goods on shelf in the retail store that will lead the customer to buy more goods and this situation creates greater demand of goods. Recently much emphasis is given to study the inventory control and maintenance of production inventories of the stocks during the storage period. For solving any inventory problems, it is highly essential for the business institutions to obtain the Economic Order Quantity (EOQ) and obtaining this quantity leads to reduce the total average inventory cost. Thus the business institutions emphasize on inventory management and solving inventory problems. The problem can only be solved if a suitable inventory model could be established that is fit for all the parameters concerned like, market demand, production rate, product's life, etc. Production-inventory model plays a dominant role in production scheduling and planning. The economic production quantity (EPQ) model is commonly used by practitioners in the fields of production and inventory management to assist them in making decision on optimum production and total cost. The EOQ assumption of instantaneous order arrival means the whole order quantity (i.e., all units of the purchased lot) is received at once from the supplier. This assumption is not valid if these units are manufactured and received over an extended time interval. The EPQ model relaxes this assumption by incorporating a gradual order receipt, i.e. a finite production rate. Demand variability according to item availability and price is a frequently observed phenomenon. In stock-dependent demand models, the demand for a given item increases with higher item availability. A bigger item display tends to attract the attention of more customers, leading to increased sales. Customers may view larger item stock as an indication of the item's popularity and also as a sign of reliable, continuing supply.

Generally, there are two kinds of materials in the daily needs as far as damage, wastage, deterioration or decay is concerned. Items like radioactive substances, food grains, fashionable items, pharmaceuticals, etc. are the items of finite life and the items like electronic goods, steels, woods, etc. are the items of long life. Due to the limited shelf-life and market demand, the stock level or inventory continuously decreases and the items in the inventory deplete or deteriorate. This deterioration affects the inventory and inventory cost increases. To make the inventory cost at optimum level, i.e. to get the minimum inventory cost, a suitable inventory model is required to be formulated. The models contain the parameters to be estimated rather than assuming it to be constant. The parameters can be estimated by using different methods like MLE, Baye's, etc. The Bayesian estimation is a framework for the formulation of statistical inference problems. In the prediction or estimation of a random process from a related observation signal, the Bayesian philosophy is based on combining the evidence contained in the signal with prior knowledge of the probability distribution of the process. Estimation theory is concerned with the determination of the best estimate of an unknown parameter.

There are several research papers available in the literature that have studied the inventory problems of deteriorating items for continuous and periodic review production models. A production lot size inventory model for deteriorating items and arbitrary production and demand rate is established by Balkhi et al. (1996). Mondal, Bhunia, and Maiti (2003) have established an inventory system of ameliorating items for price dependent demand rate. Pal (1990) has proposed an inventory model for deteriorating items when demand is random. Subbiah, Rao, and Sathyanarayana (2004) have been reviewed an inventory models for perishable items having demand rate dependent on stock level. Mishra (2013) has focused on an inventory model of instantaneous deteriorating items with controllable deteriorating rate for time dependent demand and holding cost. Roy (2008) has considered an inventory model for deteriorating items with price dependent demand and time varying holding cost. Singh, Tripathi, and Mishra (2013) have given an inventory model with deteriorating items and time dependent holding cost. Liao (2008) has developed an EOQ model with non-instantaneous receipt and exponential deteriorating items under two-level trade credit. He, Wang, and Lai (2010) deal with an optimal production inventory model for deteriorating items with multiple market demand. All the above models assume that the demand is known and deterministic.

In this paper, a Markovian inventory periodic review production model with inter-demand time as exponential distribution is considered. The model is developed on the basis of constant production rate and constant rate of deteriorating items without shortages. The unit production cost is inversely proportional to the demand rate. The model contains the exponential parameter which is unknown and is estimated through MLE and Baye's under a squared error loss function. The conjugate Gamma prior is used as the prior distribution of exponential distribution. Finally, a numerical MCMC simulation is used to compare the estimators obtained with Expected risk and are shown graphically. The objective of the paper is to develop an optimum policy that minimizes the total average cost by using the above estimates of the parameter. The sensitivity analysis is also carried out for the model with percentage change in the parameters.

Assumptions and Notations

The following assumptions are used to develop mathematical model:

1. A single item is considered in the inventory system.
2. The inter demand time is assumed to be exponential distribution with pdf $R(t) = \lambda e^{-\lambda t}$ at any time t ≥ 0.
3. The production rate, say K= r R(t), where r >1.
4. The unit production cost is inversely proportional to the time dependent demand rate. Items deteriorate with rate θ, $0 \leq \theta \leq 1$.
5. The deteriorated units can neither be repaired nor replaced during the cycle time.
6. Shortages are not allowed.

The following are notations used in this model:

r : Rate of production

$D = \frac{1}{\lambda}$: Expected rate of demand

c_1 : Production cost per unit time

c_2 : Holding cost per unit time

c_3 : Deterioration cost per unit time

TC : The total average cost

The unit production cost v is inversely related to the expected demand rate as $c_1 = \alpha_1 D^{-\gamma}$, *where* $\alpha_1 > 0, \gamma > 0$ *and* $\gamma = 1, \gamma \neq 2$. α_1 is positive as c_1 and D are both non-negative; also higher demands result in lower unit cost of production. Therefore, c_1 and D are inversely related.

γ must be positive.

$$\frac{dc_1}{dD} = -\alpha_1 \gamma D^{-(\gamma+1)} < 0 \text{ and } \frac{d^2 c_1}{dD^2} = \alpha_1 \gamma (\gamma+1) D^{-(\gamma+2)} > 0.$$

Therefore marginal unit cost of production is an increasing function of D. Thus these results imply that, as the demand rate increases at an increasing time, the unit cost of production decreases. For this reason the manufacturer is encouraged to produce more as the demand for the item increases. The necessity of the restriction γ=1, $\gamma \neq 2$ arises the nature of solution of problem. The following figure represents the instantaneous state of inventory level.

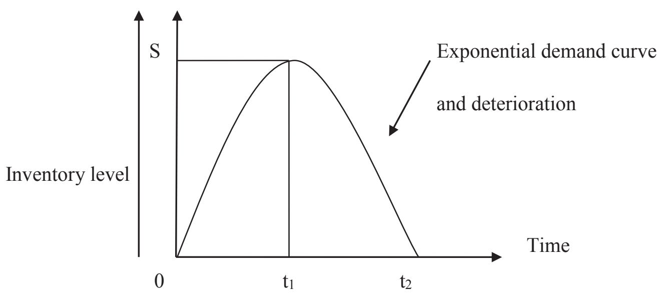

Fig. 1: Schematic Diagram Represents the Inventory Level

Description of the Model

The stock level is initially zero. Production begins just after t=0, continues up to t= t_1 and stops as soon as the stock level becomes S. Then the inventory level decreases both due to demand and deterioration, till it becomes zero at t= t_2. Then the cycle repeats itself. Let *I* (*t*) be the inventory level of the system at any time t ($0 \leq t \leq t_2$). Therefore, the inventory is described by the system of the differential equations:

$$\frac{d\,I(t)}{dt}+\theta(t)\,I(t)= K-R(t), \quad 0 \leq t \leq t_1 \;-----(1)$$

$$\frac{d\,I(t)}{dt}+\theta(t)\,I(t)=-R(t), \quad t_1 \leq t \leq t_2 \;------(2)$$

$$where\; R(t)=\lambda e^{-\lambda t}.$$

Using the value of R (t), equations (1) and (2) becomes respectively

$$\frac{d\,I(t)}{dt}+\;\theta(t)\,I(t)=(r-1)\;(\;\lambda e^{-\lambda t}), \quad 0 \leq t \leq t_1 \;-----(3)$$

with the conditions I(0) =0 and I(t_1) =S and

$$\frac{d\,I(t)}{dt}+\;\theta(t)\,I(t)=-\;\lambda e^{-\lambda t}\;,\; t_1 \leq t \leq t_2 \;------(4)$$

$$I(t)=(r-1)\left[\lambda t+\frac{\lambda\theta t^2}{2}-\lambda\theta t^2\right]+c(1-\theta t)$$

$$0 \leq t \leq t_1 ----(5)$$

Using the initial condition $t = 0$, $I(t) = 0$ in the above equation (5) the value of $c=0$

Neglecting powers of α higher than 1. This approximation is followed throughout the subsequent calculations. The solution of (4) using the condition $I(t_1) = S$ is

$$I(t)=S\left[1+\theta(t_1-t)\right]+\lambda(t_1-t)+\frac{\lambda\theta}{2}\left[t_1^2-t^2\right]+\lambda\theta\left[t^2-t\,t_1\right]$$

$$t_1 \leq t \leq t_2 ----(6)$$

For finding value of S, using the boundary condition $I(t_2)=0$, equation (6) gives

$$S\left[1+\theta(t_1-t_2)\right]=\lambda(t_2-t_1)+\frac{\lambda\theta}{2}\left[t_2^2-t_1^2\right]+\lambda\theta\left[t_1 t_2-t_2^2\right]$$

For a first-order approximation over θ, this relation gives

$$S=\lambda(t_2-t_1)+\frac{\lambda\theta}{2}\left(t_2^2-t_1^2\right)-\lambda\theta\left(t_1 t_2-t_1^2\right)----(7)$$

$$I(t)=\begin{cases}(r-1)\left[\lambda t+\frac{\lambda\theta t^2}{2}-\lambda\theta t^2\right]+c(1-\theta t) & if \quad 0 \leq t \leq t_1\\ S\left[1+\theta(t_1-t)\right]+\lambda(t_1-t)+\frac{\lambda\theta}{2}\left[t_1^2-t^2\right]+\lambda\theta\left[t^2-t\,t_1\right] & if\; t_1 \leq t \leq t_2\end{cases} ----(8)$$

Inventory during t = 0 to t_1 is

$$I_1=\int_0^{t_1} I(t)\;dt$$

$$I_1=\frac{\lambda t_1^2}{2}[r-1]+\frac{\lambda\theta t_1^3}{6}[r-1]-\frac{\lambda\theta t_1^3}{3}[r-1]----(9)$$

Inventory during t = t_1 to t_2 is

$$I_2=\int_{t_1}^{t_2} I(t)\;dt$$

$$I_2=\frac{\lambda\theta}{6}[t_2^3-t_1^3]-\frac{\lambda}{2}[t_2^2-t_1^2]+\frac{\lambda\theta}{2}[t_1^2\,t_2-t_1\,t_2^2]+\lambda[t_2^2-t_1\,t_2]----(10)$$

for a first-order approximation over α. The total number of deteriorated items [0, t_2] is given by = production in [0, t_1] – demand in [0, t_2]

$$\int_0^{t_1} r\,\lambda e^{-\lambda t}\,dt-\int_0^{t_2}\lambda e^{-\lambda t}\,dt \;=\; r\,\lambda t_1-\lambda t_2----(11)$$

Production cost in [0, t_1] is

$$c_1\left[\int_0^{t_1} r\,\lambda\,t\,dt\right] \;=\; c_1\left[r\,\lambda\frac{t_1^2}{2}\right]------(12)$$

The total average total cost (TC) is given by

TC = Production cost + Inventory cost + Deteriorating cost

$$TC=\frac{1}{t_2}\left\{c_1\left[\frac{r\lambda t_1^2}{2}\right]+c_2\left[(r-1)\left(\frac{\lambda t_1^2}{2}+\frac{\lambda\theta t_1^3}{6}-\frac{\lambda\theta t_1^3}{3}\right)+\frac{\lambda\theta}{6}[t_2^3-t_1^3]\right.\right.$$
$$\left.\left.-\frac{\lambda}{2}[t_2^2-t_1^2]+\frac{\lambda\theta}{2}[t_1^2\,t_2-t_1\,t_2^2]+\lambda[t_2^2-t_1\,t_2]\right]+c_3\left[\,r\lambda t_1-\lambda t_2\,\right]\right\}----(13)$$

Using calculus to minimize the Average total cost for finding the optimum values of t_1 and t_2 are the solutions of the equation (13) gives

$$\frac{\partial TC}{\partial t_1}=0 \quad and \quad \frac{\partial TC}{\partial t_2}=0 ------(14)$$

Provided that they satisfy the sufficient conditions

$$\frac{\partial^2 TC}{\partial t_1^2}>0,\ \frac{\partial^2 TC}{\partial t_2^2}>0 \ and \left(\frac{\partial^2 TC}{\partial t_1^2}\right)\left(\frac{\partial^2 TC}{\partial t_2^2}\right)-\left(\frac{\partial^2 TC}{\partial t_1 \partial t_2}\right)^2>0$$

Equations (14) can be written as

$$c_1\left[\frac{r\lambda t_1}{t_2}\right]+c_2\left[\frac{r\lambda t_1}{t_2}+\frac{r\theta\lambda t_1^2}{2t_2}-\frac{r\theta\lambda t_1^2}{t_2}+\lambda\theta t_1-\lambda-\frac{\lambda\theta t_2}{2}\right]+c_3\left[\frac{r\lambda}{t_2}\right]----(15)$$

$$-c_1\left[\frac{r\lambda t_1^2}{2t_2^2}\right]-c_2\left[\begin{array}{l}\frac{r\lambda t_1^2}{2t_2^2}-\frac{r\lambda\theta t_1^3}{6t_2^2}+\frac{r\lambda\theta t_1^3}{3t_2^2}-\frac{\lambda}{2}-\frac{\lambda\theta t_2}{3}+\frac{2\lambda\theta t_2}{3}\\ +\lambda-\frac{\lambda\theta t_1}{2}\end{array}\right]-c_3\left[\frac{r\lambda t_1}{t_2^2}\right]---(16)$$

The above equations are non-linear simultaneous equations (15) and (16) which cannot be solved directly. Moreover, the above equations also contain the unknown parameters and are found using MLE and Baye's in the subsequent sections of the paper, Also using R-Software to solve the non-linear equations the optimal time periods t_1^* and t_2^* are obtained. Then the maximum inventory level and optimal total cost are also obtained for the optimal time periods t_1^* and t_2^* from equations (7) and (13) respectively.

Parameter Estimation

Maximum Likelihood Estimation

The probability density function of the exponential distribution is given by,

$$f(x\,;\lambda)=\begin{cases}\lambda e^{-\lambda x} & if \quad x\ge 0\\ 0 & if \quad x<0\end{cases}$$

where λ is the parameter to be estimate

d. The MLE of λ given by $\dfrac{n}{\sum_{i=1}^{n} x_i}$

Bayes Estimation

In this section, we consider the Baye's estimation for the parameter λ of exponential distribution assuming the conjugate of prior distribution for λ as two parameter Gamma distribution given as

$$f(\lambda/\alpha,\beta)=\begin{cases}\frac{\beta^{\alpha}}{\gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta x} & ,\lambda\ge 0\\ 0 & ,\lambda<0\end{cases} \quad \alpha>0,\ \beta>0$$

The likelihood function is assumed as L (λ/x) and the posterior distribution is,

$$p(\lambda/x)\propto L(\lambda/x)\ f(\alpha,\beta)$$

$$p(\lambda/X)\propto\lambda^{n+\alpha-1}e^{-\lambda[\beta+\sum_{i=1}^{n}x_i]}$$

This follows Gamma distribution with parameter $\gamma(n+\tilde{\alpha}\ ,\tilde{\beta}+\sum_{i=1}^{n}x_i)$

The mean and variance are given by

$$\text{Mean}=\frac{\alpha}{\beta}=\frac{n+\tilde{\alpha}}{\tilde{\beta}+\sum_{i=1}^{n}x_i} \quad \text{Variance}=\frac{\tilde{\alpha}}{\tilde{\beta}^2}$$

Numerical Simulation

To compare the different estimators of the parameter λ of the exponential distribution, the risks under squared error loss of the estimates are considered. These estimators are obtained by maximum likelihood and Baye's methods under Expected risk. The MCMC procedure for Baye's estimation is as follows

A sample of size n is then generated from the density of the exponential distribution, which is considered to be the informative sample.

The MLE and Baye's estimators are calculated with

$$\alpha=n+\tilde{\alpha}\ ,\beta=\tilde{\beta}+\sum_{i=1}^{n}x_i$$

Steps (i) to (ii) are repeated N = 2000 times for different sample sizes and the risks under squared error loss of the estimates are computed by using:

Expected Risk $(\hat{\lambda})=\frac{1}{N}\sum_{i=1}^{n}\left(\lambda_i-\lambda\right)^2$ Where, $\hat{\lambda}_i$ is the estimate at the i[th] run

Assuming the value of λ = 0.2, the estimated value of $\hat{\lambda}$ using MLE and Baye's along with Expected risk are given in Table 1.

Table 1: Parameter Estimation of and Expected Risk

n	Criteria	$\alpha = n+\tilde{\alpha} \quad ,\beta = \tilde{\beta}+\sum_{i=1}^{n} x_i$			
		MLE	**α=0.5, β=0**	**α=1, β=0.5**	**α=1.5, β=1**
10	Estimated value	0.2731	0.2544	0.2622	0.2703
	ER	0.00008	0.00007	0.00006	0.00006
25	Estimated value	0.2455	0.2398	0.2361	0.2370
	ER	0.00006	0.00005	0.00004	0.00005
50	Estimated value	0.2421	0.2354	0.2351	0.2348
	ER	0.000008	0.000006	0.000006	0.000005
75	Estimated value	0.2211	0.2169	0.2171	0.2121
	ER	0.000004	0.000004	0.000004	0.000002
100	Estimated value	0.2010	0.1982	0.1970	0.1964
	ER	0.000002	0.000002	0.000002	0.000002
125	Estimated value	0.1962	0.1958	0.1932	0.1911
	ER	0.000002	0.000002	0.000002	0.000002
150	Estimated value	0.1910	0.1906	0.1906	0.1906
	ER	0.000002	0.000002	0.000002	0.000002

It is seen that for small sample sizes the estimators under the Expected Loss function have smaller ER when choosing proper parameters α and β. But for larger sample sizes (n>50), all the estimators have approximately same ER. The obtained results are demonstrated in Table 1 and shown graphically in Fig. 2 and 3. The estimated value of $\hat{\lambda} = 0.1906$

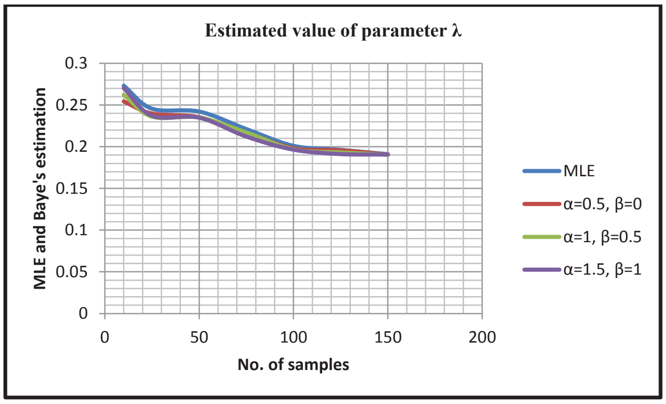

Fig. 2: MLE and Baye's Estimation

The above Fig. 2 shows the estimated value of parameter through MLE and Baye's.

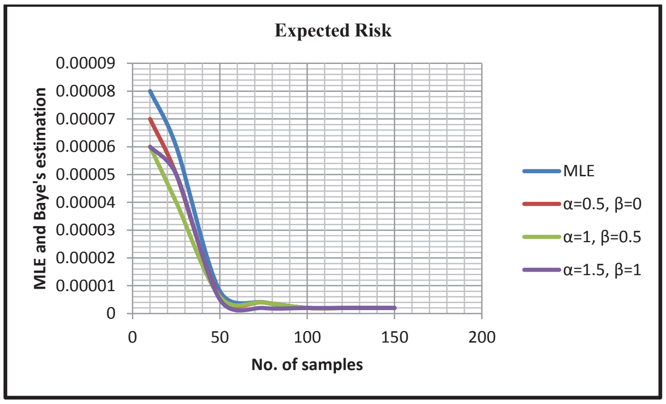

Fig. 3: Expected Risk under Loss Function

The above Fig. 3 shows the expected risk under loss function for different α and β values through MLE and Baye's.

Optimal Solutions Using Estimated Parameters

From the above MLE and Baye's estimates it is found that λ = 0.1906. For finding the optimum values we assume $\alpha_1 = 50$, $c_1 = 10$, $c_2 = 4$, $c_3 = 0.05$, $\theta = 0.01$ and r = 5 in appropriate units. Solving the non-linear equations (15) and (16) of the above model using R-software, the optimal time periods of t_1 and t_2 are obtained as $t_1^* = 77.4026$ and $t_2^* = 163.5043$ and the optimal average total cost is $TC^* = 480.9938$, the maximum inventory level is $S^* = 895.9292$.

Sensitivity Analysis

The effects of changes in the system parameter λ on the optimum values of t_1^*, t_2^* and TC^* are studied in the model. The sensitivity analysis is performed by changing each of the parameter by +50%, +25%, +10%, -10%, -25%, -50%. The results are shown in Table II.

On the basis of the results in Table 2, as the parameter λ and θ changes, the cycle time periods and total cost also changes.

Table 2: Variation of the Total Cost for Different Cycle Time Period

Parameter changes in %	Changes in λ			Changes in θ		
	t_1^*	t_2^*	TC^*	t_1^*	t_2^*	TC^*
+50%	116.1040	245.2565	1071.59	76.4115	194.3662	479.5313
+25%	96.7330	204.3374	747.5459	76.9071	178.9353	477.0616
+10%	85.1185	179.8032	580.5282	77.2044	169.6766	478.6092
-10%	69.6461	147.1195	390.1916	77.6008	157.3319	484.5318
-25%	58.0723	122.6711	272.0801	77.8982	148.0733	492.1783
-50%	38.7013	81.7521	121.4336	78.3937	132.6423	512.0902

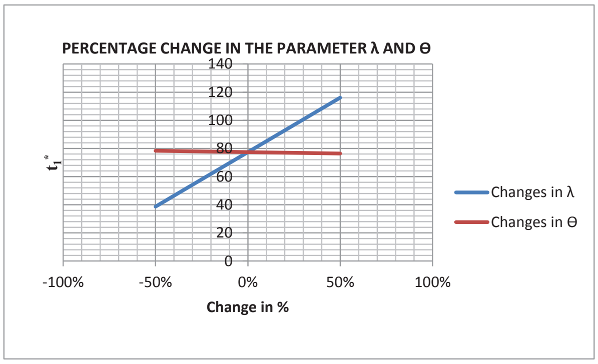

Fig. 4: Percentage Change in the Parameter λ and θ for t_1^*

Fig. 4 shows the sensitivity analysis in optimal t_1^*. It is observed that highly sensitive to changes in the parameter λ and slightly sensitive to changes in the parameter θ.

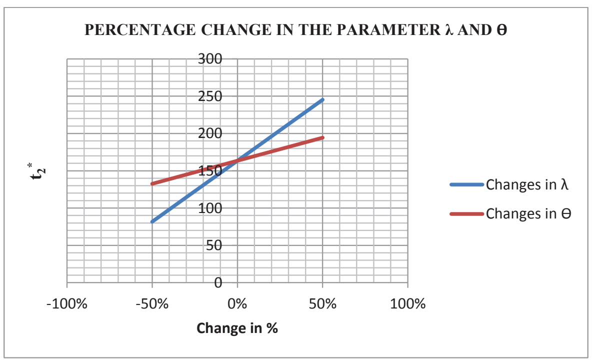

Fig. 5: Percentage Change in the Parameter λ and θ for t_2^*

Fig. 5 shows the sensitivity analysis in optimal t_2^*. It is observed that highly sensitive to changes in the parameter λ and moderately sensitive to changes in the parameter θ.

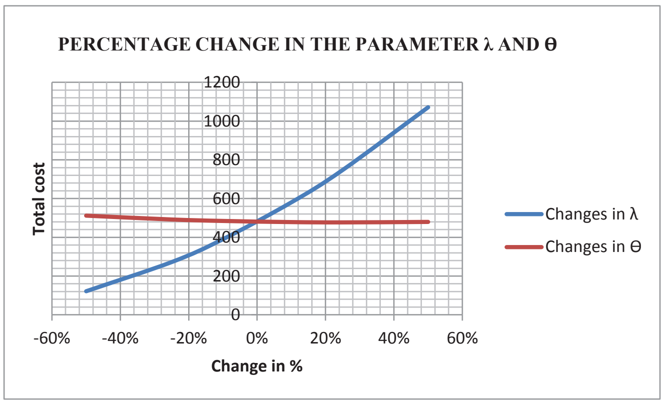

Fig. 6: Percentage Change in the Parameter λ and θ for Total Cost

Fig. 6 shows the sensitivity analysis in optimal total cost. It is observed that highly sensitive to changes in the parameter λ and slightly sensitive to changes in the parameter θ.

Conclusion

A Markovian Inventory periodic review production model with inter-demand time as exponential distribution with constant production rate for deteriorating items is considered. The model is developed on the basis of constant production rate and constant rate of deteriorating items without shortages. The unit production cost is inversely proportional to the demand rate is considered in this study. The model parameters are estimated though MLE and Baye's and R-Programme have been developed for the non-linear equations involved in the model. The exponential parameter is estimated through MLE and Baye's under a squared error loss function. The conjugate Gamma prior is used as the prior distribution of exponential distribution. Finally, a numerical MCMC simulation is used to compare the estimators obtained with Expected risk and are shown graphically. By using the estimates of the parameters involved in the model, the optimal time periods and total cost have been found as $t_1^* = 77.4026$, $t_2^* = 163.5043$, $TC^* = 480.9938$, and the maximum inventory level is $S^* = 895.9292$. Sensitivity analysis is also carried out with percentage change in the parameters.

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