Explicit Time Discretization Programming Approach to Risk Modelling

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Abstract

In this paper we formulate an explicit time discretization model for modeling risk by establishing an initial value problem as a function of time. The model is proved stable and the scaled-stability regions can encapsulated the volatile macroeconomic condition pertaining to financial risk. The model is extended to multistage schemes where we test for convergence under higher-order difference equations. Further, for addressing advection problems we have used Runge-Kutta method to propose a multistep model and have shown its stability patterns against general and absolute stability conditions. The paper also provides second-order and forth-order algorithm for computational programming of the models in practice. We conclude by stating that explicit time discretization models are stable and adequate for changing business environment.

Keywords:
Explicit time discretization, Runge-Kutta Method, algorithms, computational programming, risk modeling.

JEL Code: C31, C62, C63, G32, D81

1. Introduction

The credit risk models are primarily of two types: (i) credit pricing models and (ii) portfolio credit risk models (value-at-risk, VaR models). The VaR models have become the mainstream technique to mitigate financial risk in leading financial systems. They act as the risk management technique in to zero-down financial risks related to investment and business processes across the globe. The model has to encapsulate various instances of risk, i.e. “spread risk”, “default risk”, downgrade risk” and concentration risk” Therefore, since long the users of the VaR have periodically questioned and tested the reliability and consistency of the models in predicting and evaluating risk.

There have been issues related to the modelling of VaR related to the above mentioned aspects. First, market-risk and credit risk are both subjected to spread risk which is influenced by the volatility of the capital markets. Hence it affects the credit spreads in turn changing the credit ratings. However as downgrade risk is a credit spread risk, it can lead to exaggerated value of spread risk if it is added credit spread. Second, as the market participants forecast events and anticipate credit status before it is subjected to practise, it leads to disentangling of credit risk and market risk. Third, when the obligor is incapable of servicing any further debt, it leads to default risk. This states that default risk is a special case of downgrade risk and hence an adequate risk model (VaR) should address these issues in an integrated framework.

Finally the overall profitability of a firm is dependent of various factors including economic conditions, stock market conditions, exchange rates etc. Therefore exposures to all these factors act as instances for the measuring obligor’s default probability. Under all these circumstances the stability of the VaR models has been the major area of concern.

Backtesting of VaR has therefore been an area of research to continuously improve the adequacy of this technique during the last two decades. The backtesting of the various VaR models have given birth to new generation autoregressive VaR models. Many of these models employ aturegressive integrated moving average (ARIMA) modelling for integrating the ex-ante VaR results with the backtesting results for forecasting return distributions with higher accuracy.

In this regard the paper formulates an explicit time discretization model (ETDM) and proves its stability. The model is further extended to multistage schemes for its application in various market environments.

The rest of the paper unfolds as follows: Section 2 gives the literature review on risk modelling. Section 3 develops the explicit time discretization model (ETDM) for risk modelling and proves the stability of the models. Section 4 provides a brief discussion on putting the ETDM in practice and finally section 5 concludes the study.

2. Literature Review

Cox and Snell (1989) and Lewis (1980) devised methods to binary time-series which are auto-correlated using two-state Markov chain, binary auto-regressive moving average (ARMA) and Bayesian models respectively. Stochastic optimization technique was introduced by Fogel et.al. (1966). The approach was efficient in cases where assumptions could not be made on the one-dimension time series. These models could work in both stationary and non-stationary data set but the results experienced low explanatory powers on the regressed equations. Hyndman (1999), Muenz and Rubinstein (1985), Czado and Song (2001) introduced various models to capture distributional conformity as a function of systematic risk factor. They used nonparametric models, two –stage Markov model, logistic regression and state space models respectively. However these models failed in multi-dimensional data set operating on a single confidence interval.

As a solution to the above mentioned deficiencies in risk modeling, Leisch et.al. (1998) proposed a model where he first performed cluster analysis on the binary data-set and then used cluster identifiers for further analysis. Modeling of multivariate dataset as a function of systematic risk factor for a single portfolio was further studied by Carey et. al. (1993) using logistic regression and Chib and Greenberg (1998) using probit model. Goldstein (1999) developed mixed response models for the analysis of two binary and interval measures in a multilevel framework.

The backtesting of VAR for 'n' number of portfolios at a stated confidence interval can be done by comparing the VAR measures of each portfolio with their corresponding time-series returns at time 't'. Cameron and Trivedi (1998) created an n-dimensional vector and analyzed the summation of its elements. Jung and Tremayne (2001) tested the presence of serial-correlation error whereas Cox and Lewis (1996) investigated the issue of stationarity on the matrix formed. Jung and Treyman (2000) formulate integer-valued autoregressive moving averages for correlated dataset. The Poisson autoregressive model and the Poisson exponential weighted moving average model developed by Brandt and Williams (2000 and 1998) respectively can be used for the determining distributional conformity. For extention of VAR to multiple confidence level Chib and Winklemann (2000) formulated Markov-chain Monte Carlo Model. On similar lines King (1998) formulated the Poisson regression model and Jorgenson et.al. (1998) introduced state-space model.

Hardle and Chen (1994) introduced smooth-transition autoregressive models and exponential autoregressive model as a class of parametric conditional heteroscedastic models to determine the mean and volatility parameters of the time series data. A series of non-parametric function coefficient models were also formulated by them to measure the mean and the volatility components.

However in all the models developed in the area of measuring risk the stability factor of these models are not brought to concern. There are different models for addressing different components of the multivariate longitudinal time series data set analyzing the fixed and the random effects. This paper formulates an explicit time discretization model for measuring risk as a variant of time and states the stability of the model in the following section.

3. Explicit Time Discretization Model Development

In this section we discretize the time variables in a time-dependent differential equation by applying finite difference operations. For this we consider an initial condition defined by $f(t = 0) = f_0$ where 't' is time. For a given function 'G', if $G(f)$is independent of 't', unlike risk modeling, the equation becomes an autonomous ordinary equation which can be solved using efficient algorithms. For the purpose of modeling risk we consider an initial value problem (IVP) as:

$$\frac{df}{dt} = G(f,t) \tag{3.1}$$

3.1 Multistep Schemes

The discrete analog of equation (3.1) has several time steps involved and hence we represent it as:

$$\frac{f^{n+1} - f^{n-k}}{(k+1)\Delta t}\Big|_{k\geq 0} = G(t, f^{n+1}, f^{n}, f^{n-1}, ...) \tag{3.2}$$

Where, $f^n \equiv f(n\Delta t)$. In the multistep scheme the function depends on previous time levels of the solutions. The Euler-forward can be represented as the one-sided finite difference of the equation (3.3). The equation is defined as:

$$\frac{f^{n+1} - f^{n}}{\Delta t} = G(t, f^{n}) \Rightarrow f^{n+1} = f^{n} + \Delta t G(t, f^{n}) \tag{3.3}$$

For the second-order accuracy the eigenvalue model derived from the equation (3.2) is:

$$\frac{f_{n+1} - f_{n-1}}{2\Delta t} = \lambda f^{n} \tag{3.4}$$

For higher-order accuracy, the generalized k-step scheme can be represented as:

$$\sum_{j=0}^{k} \alpha_j f^{n+1-j} = \Delta t \sum_{j=0}^{k} \beta_j G^{n+1-j} \tag{3.5}$$

Where, $\alpha_0 = 1$ is taken for normalization and $\beta_0 = 0$ for the explicit scheme.

3.2 Consistency and Stability: Convergence

To prove that the multistep scheme is consistent we consider an equivalent differential equation for the IVP represented by the equation (3.1) for the Euler-forward scheme. Applying Tayor's expansion we get:

$$\left(\frac{df^{n}}{dt} - G^{n}\right) = \Delta t\left(-\frac{1}{2!}\frac{d^{2}f^{n}}{dt^{2}}\right) + \Delta t^{2}\left(-\frac{1}{3!}\frac{d^{3}f^{n}}{dt^{3}}\right) + ... \tag{3.6}$$

In the above equation as , the right hand side of the equation takes the value zero, reducing the equation to its original initial value problem. Therefore, the multistep scheme satisfies the consistency requirement.

In boundary value problems (BVP), consistency is sufficient to prove convergence, however in case of IVP the equation has to be tested for stability. The figure below explains the need for stability testing for IVP.

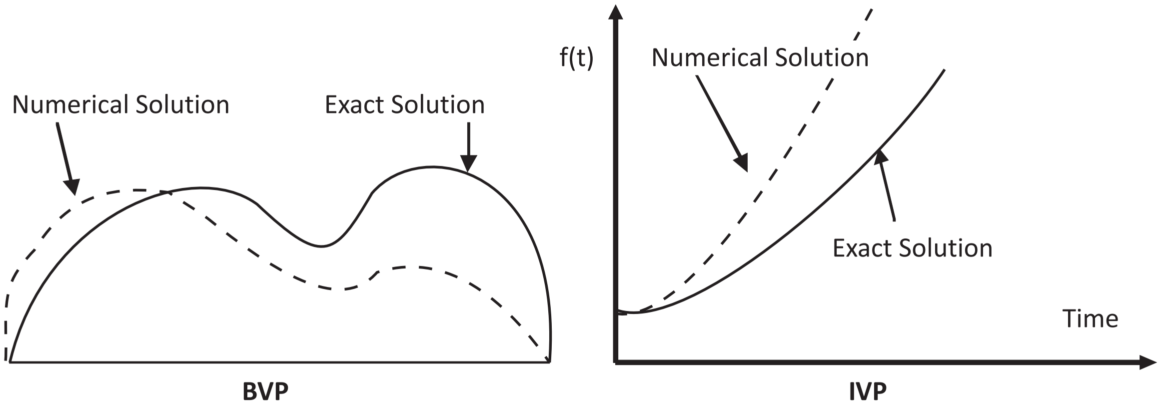

Figure 3.1 Comparison of Boundary Value Problems (BVP) and Initial Value Problems (IVP)

In the IVP there is an accumulation of error over time integration which may lead to instability in results. To prove for stability we define the error as:

$\varepsilon \equiv f_{ex}^{n} - f^{n}$

Based on the above equation we define the following:

(a) General Stability condition:

$\varepsilon \equiv f_{ex}^{n} - f^{n}$

Where $t_n = n\Delta t$. The above equation states that errors are restricted by a function dependent on time at a fixed time $(t_n = fixed, \Delta t \to 0, n \to \infty)$.

(b) Absolute Stability condition:

$$\left|\varepsilon^{n+1}\right| \le \left|\varepsilon^{n}\right|$$

For absolute stability, all the constituents of errors have to be uniformly bounded. Therefore in the above equation inequality states strong-form of stability and equality implies weak-form of stability.

To examine the stability of the equation (3.1) we establish a corresponding eigenvalue model problem defining λ as a complex number.

$$\frac{df}{dt} = \lambda f, \lambda \in C, \mathrm{Re}(\lambda) < 0 \qquad (3.10)$$

Here the error satisfies: $\varepsilon^{n+1} = \varepsilon^{n}(1+\lambda\Delta t)$ where we have:

$\left|1+\lambda\Delta t\right| < \left|1-\lambda\Delta t\right| < e^{-\lambda\Delta t}$Thus we obtain:

$$\left|e(n+1)\right| < e^{-\lambda\Delta t}\left|\varepsilon^{n}\right| \le (e^{-\lambda\Delta t})^{n}\left|e^{0}\right| \Rightarrow \left|\varepsilon\right| < e^{-\lambda(n\Delta t)}\left|\varepsilon^{1}\right| \qquad (3.11)$$

Under the general stability condition the above equation stands stable but considering an IVP which is time dependent as in case of risk modeling. In case of extreme conditions of risk modeling where IVP may be defined as:

$$\frac{df}{dt} = -f, f(0) = 1, f_{ex} = e^{-t}$$

In the IVP above at $\Delta t = 3$ we get $f^{n+1} = -2f^{n}$ and generate a sequence (1, –2, 4, –8.16...) which doest not converge at . Hence in such a case the general stability condition does not hold good. However the IVP represented by the equation can be stated stable by applying absolute stability condition. To do this we bound the error by imposing:

$\left|1+\lambda\Delta t\right| \le 1 \Rightarrow \Delta t \le -2/\lambda$ This condition is violated when we take . Therefore in case of unusual circumstances where the IVP has to be altered, the difference equation can be proved stable by imposing boundary conditions under absolute stability framework.

3.3 Stability Regions and Absolute Stability

We consider a linear ordinary differential equation to define the risk model. The model defines as a complex number and is represented as:

$$\frac{df}{dt} = \lambda f + q(t), \lambda \in C, \mathrm{Re}(\lambda) < 0 \qquad (3.13)$$

Independent of the non-homogeneity of $q(t)$ the error ε_n at time interval $(n\Delta t)$ satisfies the equation:

$$\sum_{j=0}^{k}\left(\alpha_j - \Delta t\lambda\beta_j\right)\varepsilon^{k-j} = 0$$

For computing the above difference equation we consider $\varepsilon^n = z^n$ and hence we get:

$$\sum_{j=0}^{k}\left(\alpha_j - \Delta t\lambda\beta_j\right)z^{k-j} = 0$$

In the above equation we introduce two polynomials defined as:

$$\rho(z) = \sum_{j=0}^{k}\alpha_j z^{k-j} \quad \text{and} \quad \sigma(z) = \sum_{j=0}^{k}\beta_j z^{k-j} \qquad (3.16)$$

Substituting the values of the polynomials from equation (3.16) to equation (3.15) we get ‘z’ as the roots of the polynomial defined by:

$$\Pi(z) = \rho(z) - \lambda\Delta t.\sigma(z) = 0 \qquad (3.17)$$

Assuming k distinct roots, the generalized solution for becomes:

$$\varepsilon^{n} = \sum_{j=1}^{k} c_j z_j^{n} \qquad (3.18)$$

But, the term

$\left(c_j + c_{j+1}n + c_{j+2}n^2 + \ldots + c_{j+m-1}n^{m-1}\right)z_j^{m}$ is present for m-folded roots. Therefore the errors amplify irrespective of Δt for any $\left|z_j\right| \ge 1$. This makes the corresponding scheme unstable.

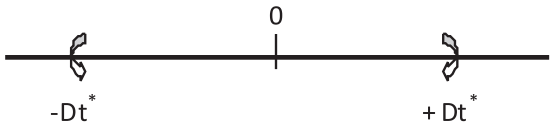

Figure 3.2 Sketch showing neighborhood around ‘0’

However we have $z_j = z_j(\lambda\Delta t)$, i.e. the roots depend on Δt. Therefore, considering $\left|z_j(0)\right| < 1 \Rightarrow \exists\Delta t^{*} : \left|z_j(\lambda\Delta t)\right| \le 1$ in the boundaries around ‘0’ with radius as as shown the figure above we get the Taylor series expansion of $z_j = z_j(\lambda\Delta t)$ as:

$$z_j(\lambda\Delta t) = z_j(0) + \varphi(\lambda\Delta t) \qquad (3.19)$$

Thus we obtain:

$$\left|z_j^n(\lambda\Delta t)\right| \le \left|z_j(0) + C\Delta t\right|^n \le \left|1 + C\Delta t\right|^n \le e^{Cn\Delta t} \le e^{Ct} \qquad (3.20)$$

Therefore considering $\left|z_j(0)\right| \le 1$, we get the roots of equation (3.17) less than unity which implies stability.

To determine the constraints representing the regions of strong absolute stability we find the roots defined by: $1\Delta t : \left|zj(1\Delta t)\right| \le 1$. For this purpose we construct a Euler-forward polynomial of the form:

$$\Pi_{EF} = (z-1) - \lambda\Delta t.1 = z - (1 + \lambda\Delta t)$$

The roots obtained for the above equation is $z = 1 + \lambda\Delta t$. To have stability we require:

$$-1 \le 1 + \lambda t \le 1 \Rightarrow \left|\lambda\Delta t - (-1)\right| \le 1 \Rightarrow \Delta t \le \frac{-2}{\lambda}, \lambda \in C, \mathrm{Re}(\lambda) < 0 \qquad (3.22)$$

The regions of stability are discussed in the next section. In general to obtain stability in multistep schemes we construct the curve with $z = e^{i\theta}, \theta \in [0, 2\pi]$ defined by the equation:

$$\lambda\Delta t = \frac{\rho(z)}{\sigma(z)}$$

3.4 Applying Runge-Kutta Methods

Appluing this method we equate the weighted sum of corrections Δf^k to the solutions at the multistep scheme to obtain:

$$f_{n+1} = f_n + C_1\Delta f^1 + C_2\Delta f^2 + C_3\Delta f^3 + \ldots$$

The coefficients C_K of the above equation are determined by comparing this equation with the Taylor series expansion. This method can be further extended for higher order computation.

$$\Delta f^1 = \Delta t G^n(t^n, f^n) : Euler - forward$$

$$\Delta f^2 = \Delta t G\left(t^n + \alpha\Delta t, f^n + \beta\Delta f^1\right)$$

In the equation other parameters are introduced to accommodate multistage schemes.

3.4.1 Stability of Runge-Kutta Methods

To check the stability of the Runge-Kutta models we consider the following IVP.

$$\frac{df}{dt} = \lambda f$$

The Runge-Kutta equations are computed for the IVP:

$$\begin{aligned} &X_1 = \lambda f^n, \\ &X2 = \lambda\left(f^n + \frac{1}{2}\lambda f^n \Delta t\right) \\ &\vdots \end{aligned} \qquad (3.28)$$

Therefore we get:

$$f^{n+1} = f^n + \frac{1}{6}\Delta t\left(X_1 + X_2 + 2X_3 + X_4\right)$$

Or $f^{n+1} = f^n\left[1 + \lambda\Delta t + \frac{\lambda^2\Delta t^2}{2!} + \frac{\lambda^3\Delta t^3}{3!} + \frac{\lambda^4\Delta t^4}{4!}\right]$ (3.29)

From the above equation we get the growth factor as:

$$G = \left[1 + \ldots + \frac{\lambda^4\Delta t^4}{4!}\right] \qquad (3.30)$$

For the stability of the Runge-Kutta model we require $|G| \le 1$. Now by setting $\mu \equiv \lambda\Delta t$ we solve to obtain $\mu(\theta)$:

$$1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^2}{3!} + \frac{\mu^4}{4!} = e^{i\theta}, \theta \in [0, 2\pi] \qquad (3.31)$$

The stability patterns are discussed in the next section and are compared with the multistep schemes.

4. Putting the Explicit Time Discretization models in practice

In this section we show graphically the stability regions of the above derived ETD models. For physical stability of the model we have considered $\lambda\Delta t \le 0$. The diagram below represents the stability region restricted to inclusion of different parameters.

The above diagram shows the stability in the complex plane for . It touches the imaginary advection axis only at one point. Hence this scheme is marginally stable for advection problems.

Even though this might be an exceptional case for risk modeling, we have proposed models based on Runge-Kutta methodology to answer this shortcoming.

The stability regions of the Runge-Kutta method models formulated in the above section is plotted in the figure (4.2). It is evident that the stability regions of these models are directly proportional to the order of the equation. Therefore for advection modeling of risk the equations formulated using Runge-Kutta method is more adequate.

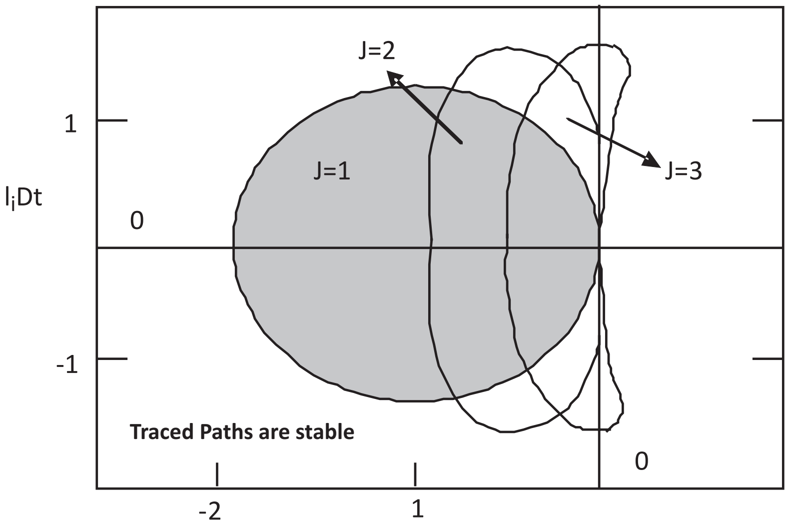

Figure 4.1 Stability diagram for Explicit Time Discretization Models

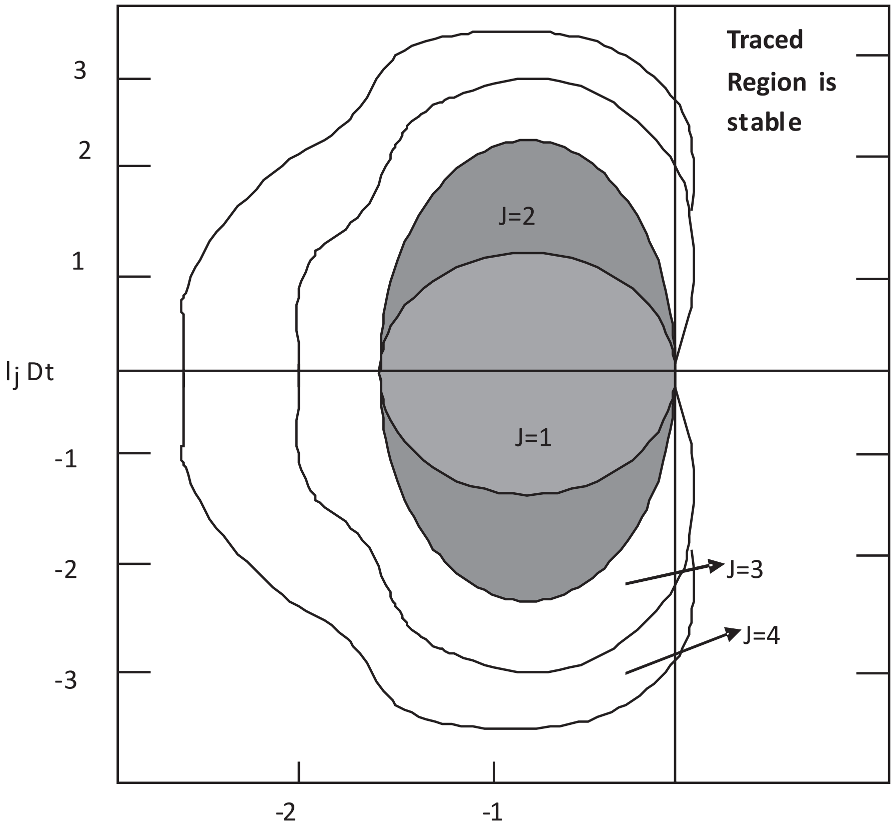

Figure 4.2 Stability diagram for Explicit Time Discretization Models (Runge-Kutta Model)

4.1 Algorithm for Second-order Runge-Kutta Method Model

Set:
$$X = f^n$$
$$Y = G(f, t^n)$$

Compute:
$$X = X + \alpha.\Delta t Y$$
$$Y = aY + G(X, t^n + \alpha\Delta t)$$

Re-run:
$$f^{n+1} = X + \frac{\Delta t}{2\alpha} Y$$

In the equation we take: $a = -1 + 2\alpha - \alpha^2$. For Euler-forward we have: $\alpha = 1/2$.

4.2 Algorithm for Forth-order Runge-Kutta Method Model

Set:
$$X = f^n$$
$$Y = G(f, t^n)$$

Compute:
$$X_1 = G(f^n, t^n)$$
$$X_2 = G(f^n + \frac{1}{2} X_1 \Delta t, t^n + \frac{1}{2}\Delta t)$$
$$X_3 = G(f^n + \frac{1}{2} X_2 \Delta t, t^n + \frac{1}{2}\Delta t)$$
$$X_4 = G(f^n + \frac{1}{2} X_3 \Delta t, t^n + \frac{1}{2}\Delta t)$$

Re-run: $f^{n+1} = f^n + \frac{1}{6}\Delta t\left[X_1 + 2X_2 + 2X_3 + X_4\right]$

5. Conclusion

This paper focuses on the modeling of risk by employing explicit time discretization on an initial value problem as a function of time. The model is extended to multistage schemes consistent to the volatility of macroeconomic factors pertaining to financial risk. The stability of explicit time discretization programming approach is studied under general and absolute stable schemes. We conclude that general stability is consistent for short term integration. The paper further explores the initial value problem employing Runge-Kutta methods for advection scenarios. The convergence of these models is also shown through scaled-stability regions. To conclude we state that the Runge-Kutta based models are more stale for advection programming however the ETD models can be used for risk modeling where the chances of advection problem is seldom. We also provide algorithms of second and forth-order for computational programming of the formulated risk models.

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