REDUCED FEASIBLE REGION AND BOUNDS FOR IMPROVED EFFICIENCY OF THE BRANCH AND BOUND METHOD OF LINEAR INTEGER PROGRAMMING

Elias Munapo
Department of Decision Sciences, School of Economic Sciences,
College of Economics and Management Sciences,
University of South Africa, PO Box 392, Pretoria, South Africa.

Santosh Kumar
Department of Mathematics and Statistics
University of Melbourne, Parkville, Melbourne, Australia

Lutfar Khan
School of Engineering and Science, Victoria University,
PO Box 14428, Melbourne City, MC 8001, Australia.

Abstract :
This paper develops a strategy to reduce the feasible space containing the optimal integer solution of a linear integer-programming model before applying the branch and bound method of integer programming. The strategy improves efficiency of the branch and bound method. It also calculates upper and lower bounds for the optimal solution. A ratio using a factor of 1.5 has been used to calculate the reduction in the objective function, with the hope that the reduced feasible space contains the optimal solution. This ratio of 1.5 has been tested on 200 randomly generated integer models. The procedure of decreasing the objective Z value also helps in computing the variable ranges. With the use of variable ranges and convex feasible space reduction, the optimal integer solution can be searched efficiently by using the branch and bound approach.

Keywords: ratio, variable range, branch and bound algorithm and alternate optimal solution.

1. Introduction

The linear integer programming model (LIP) is in a class of difficult problems that has attracted attention of mathematicians for the past half a century Dakin [1], Taha [2] and Winston[3]. The available methods for solving the integer-programming problem have used ideas of cuts, branch and bound (BB), hybrids processes, Lagrange method, random search method and many other search procedures Kumar and Munapo [4]. Characteristic equation approach was developed recently to solve a LIP model by Kumar, Munapo and Jones [5]. In this investigation integer values of the non-basic variables were used to find the required integer solution. The characteristic equation is responsible for descend of the objective function in a controlled way such that the first feasible integer solution becomes an optimal solution. All these approaches are intended to get to an integer point from the LP optimal solution with controlled reduction in the objective function value obtained by the LP approach. The branch and bound idea has long been exploited for this journey form the LP optimal solution to an integer optimal solution [2, 3]. The main concern in application of the BB approach is that if one is not careful, the process may end up searching over a large region resulting in a large number of searches. Various strategies to control the number of possibilities have been discussed in Munapo et al. [6].

This paper aims to reduce the feasible region of search to contain the optimal integer solution by estimating the bounds on the variables. The problem is modified first by introducing the bounds before it is solved by the B and B method. A significant reduction has been observed in the number of sub-problems created by the BB searches. The reduction in the feasible space containing an optimal integer solution is based on a ratio of the reduction in the value of the objective function and that of a basic variable. The ratio 1.5 used in this paper has been experimented for over 200 randomly generated LIP models to obtain a decrease in the value of the objective function. The approach is suited for parallel computing as several ratios can be tested in parallel. The proposed method has consistently proved to be efficient and accurate for LIP models.

The paper has been organized in 6 sections. The mathematical background and justification has been discussed in Section 2. Section 3 presents steps of the algorithm. A numerical example has been discussed in Section 4. Computational experiments on some bench marked problems are presented in Section 5 and finally the paper has been concluded in Section 6.

2. Mathematical background

2.1 The LP solution

Let a linear integer-programming (LIP) model be given by:

$$\left.\begin{array}{l} \text{Minimize } Z=\sum_j c_j x_j \\ \text{Such that } \sum_i \sum_j a_{ij} x_j \le b_i \\ \text{Here } a_{ij},\, b_i,\, c_j \text{ are constants, } x_j \ge 0 \text{ and integer.} \\ i=1,2,\ldots,m \quad and \quad j=1,2,\ldots,n \end{array}\right\} \quad (1)$$

The integer restrictions can be relaxed and the model (1) is solved as a LP. The continuous LP optimal solution obtained is presented in Table 1.

Table 1: The general continuous optimal tableau

	x_1	x_2	. . .	x_m	s_1	s_2	. . .	s_n	$r.h.s.$
Z	0	0	. . .	0	ω_1	ω_2	. . .	ω_n	Z_{opt}
x_1	1	0	. . .	0	α_{11}	α_{12}	. . .	α_{1n}	β_1
x_2	0	1	. . .	0	α_{21}	α_{22}	. . .	α_{2n}	β_2
					.				
					.				
					.				
x_m	0	0	. . .	1	α_{m1}	α_{m2}	. . .	α_{nn}	β_m

Here $\omega_1, \omega_2, \ldots, \omega_n, \beta_1, \beta_2, \ldots, \beta_m$ are non-negative constant values, whereas α_{ij} and z_{opt} may be positive or negative constant values. The basic variables have been denoted by $x_1 \;\; x_2 \;\; . \; . \; . \;\; x_m$ and the non-basic have been denoted by $s_1 \;\; s_2 \;\; . \; . \; . \;\; s_n$. It may be noted that a given variable xj and a slack variable si in (1) can be basic or non-basic. The presentation in Table 1 is just for convenience for two kinds of variables in the final solution. The basic is represented by the symbol x and the non-basic by the symbol s just for convenience of presentation.

The continuous objective value z_{opt} has to be reduced to reach the integer optimal point. In the descending search method discussed in [5], reduction was controlled to land on the optimal integer point. In this paper reduction in the value of Z is considered so that the reduced feasible region contains the optimal integer point. To start with this reduction has been approximated by an amount (ΔZ), which is equal to the fractional part of the objective function. Using this reduction in the Z value, one can estimate reduction in values of the basic variables. Also note that if LP solution is unique all basic variables have a specific fixed value, there is no room for any variable to move from its specific value. When an additional constraint parallel to the objective function is added, it will give rise to more than one solution and hence values of basic variables are no longer unique. Thus basic variables acquire a range associated with them. The range of a variable is defined as the difference between its largest and the smallest possible value. Note that any increase in the value of the non-basic variables will decrease the objective value. Thus one gets a relation joining the non-basic variables and the reduction in the value of the objective function as given by (2).

$$\omega_1 s_1 + \omega_2 s_2 + \ldots + \omega_n s_n \ge \Delta Z \qquad (2)$$

The amount of decrease in the objective function value (ΔZ) is approximated as:

$$\Delta Z = f \qquad (3)$$

Where is the fractional part in the value of the z_{opt}.

2.2 Determination of variable ranges

Adding an extra constraint, $\omega_1 s_1 + \omega_2 s_2 + \ldots + \omega_n s_n \ge \Delta Z$, in Table 1 will result in 'n' alternative optimal solutions. These 'n' number of new extreme optimal points can be determined by pivoting on $-\omega_1, -\omega_2$ and $-\omega_n$ respectively to obtain new solutions given below by relations (4).

$$\left.\begin{array}{l} x_1=\overline{\beta}_1^1, x_2=\overline{\beta}_2^1, \ldots, x_m=\overline{\beta}_m^1 \\ x_1=\overline{\beta}_1^2, x_2=\overline{\beta}_2^2, \ldots, x_m=\overline{\beta}_m^2 \\ \qquad\qquad . \\ \qquad\qquad . \\ x_1=\overline{\beta}_1^n, x_2=\overline{\beta}_2^n, \ldots, x_m=\overline{\beta}_m^n \end{array}\right\} \quad (4)$$

From the values of basic variables as given in Table 1 and relations (4), the ranges of the variables can be obtained.

Range of basic variable x_i :

$$\beta_i^l \le x_i \le \beta_i^u \qquad (5)$$

Where $\beta_i^l = \min_l [\beta_i, \overline{\beta}_i^1, \overline{\beta}_i^2, \ldots, \overline{\beta}_i^n]$ and ,

$\beta_i^u = \max_l [\beta_i, \overline{\beta}_i^1, \overline{\beta}_i^2, \ldots, \overline{\beta}_i^n] \quad i=1,2,\ldots,m.$

The ranges of the variables are the difference between the smallest and largest values i.e.

$$\beta_i^u - \beta_i^l, i=1,2,\ldots,m \qquad (6)$$

Smallest change corresponds to the variables xi given by

$$\Delta x_i = \min(\beta_i^u - \beta_i^l) \qquad (7)$$

2.3 The reduction in the LP objective value

Consider the following ratio relation used to estimate an amount of decrease in the value of the objective function for containing integer points:

$$\Delta x : \Delta Z = \Delta X_{\text{int}} : \Delta Z_{\text{int}} \qquad (8)$$

The is a change in the range of the most restricted variable and is the decrease in the value of the objective function, which may contain an integer optimal solution. The relation (8) is expressed as:

$$\frac{\Delta x}{\Delta Z} = \frac{\Delta X_{\text{int}}}{\Delta Z_{\text{int}}} \qquad (8')$$

The distance between any two consecutive integers is 1. Thus in general when the difference between any two numbers that are not necessarily integers is less than 1, finding an integer in between is uncertain. However, if it is more than 1, chances of containing an integer point will increase. Therefore, it is wise to use a value greater than 1. In this paper the value $\Delta X_{\text{int}} = 1.5$ has been suggested. This value was used to solve over 200 randomly generated LIP models solved so far. Thus (8') may be replaced by (9):

$$\Delta Z_{\text{int}} = \frac{1.5\Delta Z}{\Delta x} \qquad (9)$$

Note that the RHS is a constant quantity determined using (3) and (7). The parallel constraint (2) is developed with RHS value equal to the value given by (9) to find the range and the bounds.

2.4 Extreme points and the reduced convex region

The optimal integer value of a variable will always be within the largest and the smallest value on the extreme points provided the feasible region is a convex one.

Proof: Suppose the 'n' extreme points in (4) are given by

$$\left.\begin{aligned} P_1 &= (\bar{\beta}_1^1, \bar{\beta}_2^1, \ldots, \bar{\beta}_m^1) \\ P_2 &= (\bar{\beta}_1^2, \bar{\beta}_2^2, \ldots, \bar{\beta}_m^2) \\ &\;\;\vdots \\ P_n &= (\bar{\beta}_1^n, \bar{\beta}_2^n, \ldots, \bar{\beta}_m^n) \end{aligned}\right\} \qquad (10)$$

The above extreme points have been generated from the LP feasible optimal solution in the convex region by adding one more linear constraint to the given set of variables. Let the optimal integer point be given by:

$$P_f = (\bar{\beta}_1^f, \bar{\beta}_2^f, \ldots, \bar{\beta}_m^f) \qquad (11)$$

Since one is dealing with a convex region, it is obvious that

$$\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n P_n = P_f \qquad (12)$$

Where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are non-negative multipliers satisfying the condition (13).

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1 \qquad (13)$$

According to the theorem

$$\beta_j^l \le \beta_j^f \le \beta_j^u \text{ for all j.} \qquad (14)$$

The proof is immediate if one considers the problem:

$$\left.\begin{aligned} &\textit{Maximize } \beta_j^f = \sum_j \lambda_j \beta_j^j \\ &\textit{Such that } \sum_j \lambda_j = 1 \end{aligned}\right\} \qquad (15)$$

and

$$\left.\begin{aligned} &\textit{Minimize } \beta_j^f = \sum_j \lambda_j \beta_j^j \\ &\textit{Such that } \sum_j \lambda_j = 1 \end{aligned}\right\} \qquad (16)$$

Since there is only one constraint, only one variable is basic at the optimal solution of (15) and (16).

For the maximization model the optimal solution is obtained when

$$\beta_j^f = \max[\bar{\beta}_1^1, \bar{\beta}_1^2, \ldots, \bar{\beta}_1^n] \qquad (17)$$

Similarly when

$$\beta_j^f = \min[\bar{\beta}_1^1, \bar{\beta}_1^2, \ldots, \bar{\beta}_1^n] \qquad (18)$$

will form a solution to a minimization model.

Thus $\beta_n^l \le \beta_j^f \le \beta_n^u$ (19)

This theorem is useful as variable ranges can be determined to find the integer bound before solving the LIP model by the branch and bound algorithm. The extreme points can also be used to determine the order of branching in addition to the variable ranges. When using the branch and bound algorithm, the number of sub-problems is significantly reduced if the branching is started with the **most restricted variable**, see Munapo et al. [6].

2.5 Special Cases

2.5.1 Case 1

In many cases LP solutions may result in $\Delta x \ge 1.5$ from (7). This may happen when alternative solutions exist and the range Δx will be determined by using all the alternate solutions.

2.5.2 Case 2

After using $\Delta X_{\text{int}} = 1.5$ or the value from case 1, it is not certain that the optimal integer point will be found. In such a case ΔX_{int} is increased by 0.5 and process repeated.

3.0 The branch and bound method in presence of bound of variables

Step (i): Solve relaxed LIP model to get a continuous optimal solution.

Step (ii): Is solution unique? If unique then go to Step (iv) else go to Step (iii).

Step (iii): Compute all the optimal extreme points. Use these alternative points to find the smallest variable range Δx. If $\Delta x \geq 1.5$ then go to Step (v) else go to Step (iv).

Step (iv): Using the values of Δx and ΔZ, compute ΔZ_{int} using $\Delta Z_{\text{int}} = \dfrac{1.5\Delta Z}{\Delta x}$

Step (v): Calculate ΔZ_{int} using (9) to compute the LP extreme points and their associated variable ranges. Apply the branch and bound algorithm after incorporating in the original problem the integer bounds. If solution is feasible then stop else go to Step (vi).

Step (vi): Increase ΔX_{int} by 0.5 and return to Step (iv).

Figure 1: The ratio approach for solving the linear integer- programming problem.

4.0 Numerical illustration

Consider the following LIP model.

$$\left.\begin{array}{ll} \text{Maximize} & Z = 8x_1 + 7x_2 + 9x_3 \\ \text{Such that:} & 6x_1 + 11x_2 - 9x_3 \leq 86 \\ & 7x_1 - 8x_2 + 2x_3 \leq 43 \\ & -8x_1 + 12x_2 + 14x_3 \leq 8999 \end{array}\right\} \quad (20)$$

Where x_1, x_2 and x_3 are non-negative integers. Solving directly by automated branch and bound method on TORA software [7], it took 321 sub-problems to reach the optimal integer solution, which is given by:

$$Z_{opt} = 8326, x_1 = 210, x_2 = 304, x_3 = 502 \quad (21)$$

The same problem when solved as a LP resulted in Table 2.

Table 2: Continuous optimal tableau for the problem (20).

	x_1	x_2	x_3	s_1	s_2	s_3	$r.h.s.$
Z	0	0	0	0.75822	1.53422	0.91111	8330.26756
x_1	1	0	0	0.06044	0.11644	0.02222	210.18311
x_3	0	0	1	-0.00889	0.07111	0.05556	502.23778
x_2	0	1	0	0.05067	-0.00533	0.03333	304.09467

$$Z_{opt} = 8330.26756, x_1 = 210.18311, x_2 = 304.09467, x_3 = 502.23778 \quad (22)$$

From (22) the fractional part of the objective is given by 0.26756, which gives

$$\Delta Z = f = 0.26756 \quad (23)$$

Thus an additional constraint 0.75822 s_1 + 1.53422 s_2 + 0.91111 s_3 $\geq$ 0.26756 is added to the Table 2. The following three alternative extreme points become new optimal solutions, as given in (24).

$$\left.\begin{array}{l} Z_{opt} = 8330, x_1 = 210.16178, x_2 = 304.07679, x_3 = 502.24091 \\ Z_{opt} = 8330, x_1 = 210.16280, x_2 = 304.09560, x_3 = 502.22538 \\ Z_{opt} = 8330, x_1 = 210.17659, x_2 = 304.08458, x_3 = 502.22146 \end{array}\right\} (24)$$

Using these extreme points, one can calculate the variable ranges given by:

$$\left.\begin{array}{l} 210.16178 \leq x_1 \leq 210.18311 \\ 304.07679 \leq x_2 \leq 304.09560 \\ 502.22146 \leq x_3 \leq 502.24091 \end{array}\right\} \quad (25)$$

$$\Delta x_1 = 0.02133,\ \Delta x_2 = 0.01881,\ \Delta x_3 = 0.01945 \quad (26)$$

$$\Delta x = \min(\Delta x_1, \Delta x_2, \Delta x_3) = \min(0.02133, 0.01881, 0.01945) = 0.01881 \quad (27)$$

Ratio

$$\Delta Z_{\text{int}} = \frac{1.5\Delta Z}{\Delta x} \quad (28)$$

$$\Delta Z_{\text{int}} = \frac{1.5X0.26756}{0.01881} \quad (29)$$

$$\Delta Z_{\text{int}} = 21.336523 \quad (30)$$

For decreasing the objective value by 21.336523, the additional constraint in Table 2 becomes:

$$0.75822s_1 + 1.53422s_2 + 0.91111s_3 \geq 21.336523 \quad (31)$$

Constraint (31) will give rise to new extreme points as follows:

$$\left.\begin{array}{l} Z_{opt} = 8294.75, x_1 = 207.38, x_2 = 301.76, x_3 = 502.71 \\ Z_{opt} = 8294.75, x_1 = 207.40, x_2 = 304.32, x_3 = 500.62 \\ Z_{opt} = 8294.75, x_1 = 210.32, x_2 = 302.79, x_3 = 500.08 \end{array}\right\} \quad (32)$$

From expressions (31) and (32) the variable ranges can be calculated as

$$\left.\begin{array}{l} 207 \leq x_1 \leq 210 \\ 301 \leq x_2 \leq 304 \\ 500 \leq x_3 \leq 502 \end{array}\right\} \quad (33)$$

After adding the bounding constraints (33) to the original problem (20), the automated branch and bound method of the TORA software resulted in the optimal integer solution is two sub-problems only.

5.0 Computational Experience

This reduction in the number of sub-problems was observed on 200 randomly generated problems. At this stage an attempt was made to try the approach to problems from the MIPLIB 3.0 library [8]. Eight integer benchmark problems were selected. The software **lp_solve (version 2.2)** [9] was used to solve the benchmark models using the branch and bound algorithm. Fortran 2003 commands were used on the approach discussed in this paper. The number of sub-problems required to verify the optimal solutions are shown in Table 3 below.

6. Conclusions

1. The approach presented in this paper is effective in reducing the number of sub-problems that are required to verify optimality in a branch and bound framework. The procedure has the advantage that only a fraction of the feasible region is searched.
2. This significant reduction was achieved by

adding variable bounds in application of the B and B method. An alternative approach for reducing the number of sub-problems in the branch and bound search, see Kumar et al. [10].

3. The various extreme points can be efficiently calculated in parallel, thus making the ratio approach suitable for parallel processing, which is likely to dominate future mathematical computations.

References

Dakin, R.J., (1965), A tree search algorithm for mixed integer programming problems, The Computer Journal, Vol. 8,pp 250-255.

Taha, H.A., (2004), Operations Research: An Introduction, Pearson Educators, 7th Edition.

Winston, W.L., (2004), Operations Research Applications and Algorithms, Duxbury Press, 4th Edition.

Kumar, S.and Munapo, E., (2008), Fifty years of pure integer programming: A review of solution methods, Communicated for publication.

Kumar, S., Munapo, E. and Jones, B.C., (2007), An Integer Equation Controlled Descending Path to a Protean Pure Integer Program, Indian Journal of Mathematics Vol. 49, pp 211-237.

Munapo, E., Jones, B. and Kumar, S., (2009), strategies For solving the characteristic equation arising in the pure integer programming problem, to appear in International Journal of Mathematical Modeling, Simulation and Applications.

Taha, H.A., (2002), TORA Software, Windows Version 1.00.

http://www.ftp.caam.rice.edu/pub/people/bixby/miplib/miplib3

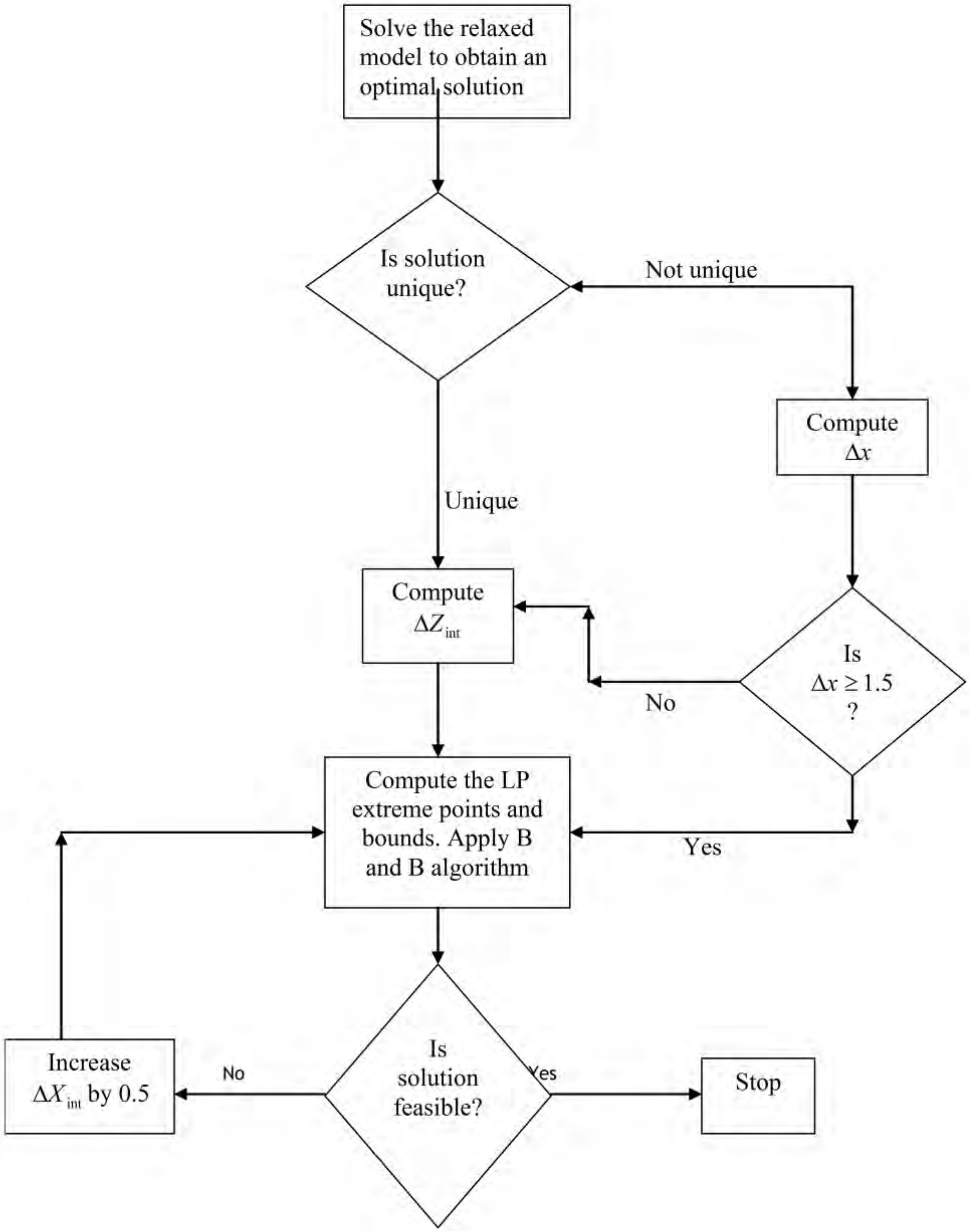
ftp://www.ftp.s.ele.tue.nl/pub/lp

Kumar, S., Munapo, E., Jones, B. and Mehlawat, M., (2008), Complexity reduction for solving pure integer programs by the branch and bound method using the Gomory constraints, ASOR Bulletin, Vol. 27, No. 2, pp. 13-22.

Table 3: Comparison of the B and B method with the conventional approach

Problem	No. of sub-problems in application of the B and B method without adding the bounds	No. of sub-problems in application of the B and B method after adding the bounds.
Enigma	9321	1847
gt2	sub-problems exploded	22948
Mod008	2848139	16756
P0033	7409	567
P0201	10247	397
P0282	sub-problems-exploded	21791
Stein27	12031	941
Stein45	235087	6875

Figure 1


Solve the relaxed model to obtain an optimal solution
Is solution unique?
Not unique
Compute Δx
Unique
Compute ΔZ_{int}
Is $\Delta x \geq 1.5$?
No
Compute the LP extreme points and bounds. Apply B and B algorithm
Yes
Is solution feasible?
Increase ΔX_{int} by 0.5
No
Yes
Stop