Linear Programming: A Practical Approach to Transportation Cost Problems

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Abstract

Cost, in accounting terms, refers to the sacrifice given up in order to obtain a product or service. Transportation cost, on the other hand, relates to the cost of carriage or conveyance of goods and services from one place (the origin – the factor) to another place (the destination – the warehouse). Various techniques and procedures are employed to minimize transportation cost. This paper dwells on the usage of linear programming approach towards solving transportation cost problems. Linear programming in a nut-shell is a term that covers a whole range of mathematical techniques that aim at optimizing performance in terms of combination of resources. It is one of the practical approaches which are employed by the cost accountant to achieve the desired objective of minimizing cost while maximizing profit and efficiency. According a brief introduction of the subject matter is proffered, and the concepts which are commonly used are examined. The paper also discusses the method of solving linear programming problems. Others areas where linear programming could be applied are also outlined.

1. Introduction

In operation research technology, programming means the use of optimizing techniques, and linear refers to the relationship between variable. Hence, linear programming models arise from he use or allocation of resources such as labour, materials, machines, capital, etc so as to optimize cost and maximize profit. Thus, the optimization effect is the objective of setting and solving the required linear programming models.

Published works on linear programming models started as early as 1939 by a Russian mathematician – L. V. Kantowvich. In 1947, George Dantiziz developed the 'simplex method' which is the most basic and systematic approach of solving linear programming problems. This paper will not give much attention to the simplex method approach of solving linear programming problems, however, a brief discussion or explanation of the terms used is offered.

Transportation cost problems can be regarded as the generalization of assignment or allocation problems. In transportation cost problem, we have m – origin with ith origin possessing ai – items and n – destination (possibly a different number from m) with j requiring bj items such that $\sum ai = \sum bj$.

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(This condition is only stated so as to enhance the smooth solution to the problem at hand; in real life situation, it is not always the case). Often times, a dummy destination is created in order to solve a particular problem.

2. Discussion

According to Dantiziz (1963), the formula of linear programming model involves the identification of the following:
(i) Decision variables
(ii) All the constraints
(iii) Objectives function

Dantiziz (1963) further stated that the problem of linear programming, in mathematical terms, is that of determining the maximum (or minimum) value of the linear function

$$f(x) = c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n \quad \text{.....................} (1)$$

called the objectives function, f, subject to the constraints (or restrictions):

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{33}x_3 + \dots + a_{2n}x_n = b_2 \\ \dots + \dots + \dots + \dots + \dots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_n \end{array}\right\} \quad \dots\dots\dots\dots\dots (2)$$

In other words, we wish to maximize (or minimize) the linear function (1) defined over a polyhedral, S, {a set of constraints or restrictions – (2)}. Each point in S is called a **feasible solution** of the problem; and any point in S which the objective function, f, takes its maximum (or minimum) value is called an **optimal solution**.

Solution of the System

Let (2) be a system of m – linear inequalities in n – unknowns. By the term 'solution of the system' we mean the set of all points in X = $\{x_{11}, x_{12}, \dots, x_{n1}\}$ whose coordinates satisfy all the inequalities simultaneously. The solution – set of linear inequalities is called a **polyhedral convex set**.

3. The term 'Solution and Solution set' in transportation problems

In ordinary day-to-day spoken English, the term solution is often understood in the sense of the final answer to a problem. However, this is not the sense in which the term is used in linear programming and its extension such as the transportation problem. Here, 'any specification of value for the decision is called a solution, regardless of whether it is desirable or even an allowable choice" (Hillier and Lieberman, 1980, p.23). From this usage, any linear programming problem has an infinite number of possible solutions which may be referred to as the **solution – set**.

Some of these solutions will satisfy the constraints while others do not. The solutions which satisfy the constraints are referred to as feasible solutions. It is possible for a problem to have no feasible solution, that is, there is no solution that can satisfy all the constraints. However, given that there are feasible solutions to a given problem, the goal of linear programming is to find which one is 'best', or the optimal solution, as measured by the value of the objectives function of the model. Thus, the optimal solution is a feasible solution that has the 'most favourable' value of the objective function. By 'most favourable' we mean the largest or smallest value, depending on whether the overall objective of the firm is that of maximization or minimization of the objective function. In other words, the optimal solution is that feasible solution which maximizes or minimizes the objective function.

4. Solving a linear programming problem using the geometrical method

The geometrical or graphical method is one of the practical approaches of solving a linear programming problem. To find the maximum (or minimum) value of a linear function – $f(x_1, x_2) = ax_1 + bx_2 = c$ over a convex polygon, the following steps are involved:
(i) Sketch the convex polygon that represents the set of feasible solutions
(ii) Graph the line $f(x_1, x_2) = ax_1 + bx_2 = c$ for some convenient value of $f(x_1, x_2)$
(iii) Use a straight edge to sketch the line in step (ii) moving the straight line in the direction of increasing (or decreasing) value of x_1, x_2
(iv) Locate the least point of the convex polygon that intersects one of the lines in step (iii)
(v) The co-ordinates of the points in step (iv) represents the value of x and y for which the function, $f(x_1, x_2)$ is a maximum (or minimum)

Below is a summary table of a research conducted by a dietician at the General Hospital, Ogoja in Cross River State of Nigeria to determine the monthly dietary / nutritional requirements of beans and plantain foods that are recommended for diabetic patients. The objective of the research was to determine the best combination of these foods that can be obtained at minimal cost while meeting the standard dietary requirements of the diabetic patient.

The dietician wishes to serve foods that provide the necessary calories, vitamins and minerals. The following chart indicates the amount of calories, vitamins and minerals to be provided and the amount (in kg) of beans food, f_1 and plantain food, f_2

Nutritional requirements	Beans food, f_1 (Units per kg)	Plantain food, f_2 (Units per kg)	Units required
Calories	3	4	340
Vitamins	1	1	100
Minerals	1	5	150

Source: Dietician Quarterly Magazine, Vol. 3, No. 12, 2003

The cost of beans food, f_1 and plantain food, f_2 is N_3 and N_5 respectively, per a kilogram weight, and the dietician wishes to minimize the total cost of the meals purchased subject to the dietary requirement shown in the table. In other words, how many kg of each food should he purchase in order to meet the dietary requirements at the least cost ∑

The solution to the above problem requires the determination of the possible combination of foods that can be used to meet the dietary requirements; and from amongst these combinations, a choice of that combination of foods that costs less. Here, the problem of meeting the dietary requirements is treated as that of minimization of cost

Let x_1 be the number of kg of beans food, f_1 and x_2 to be the number of kg of plantain food, f_2

Then the number of unit of calories, vitamin and minerals obtained will, respectively, be:

$$3x_1 + 4x_2 = 340$$
$$x_1 + x_2 = 100$$
$$x_1 + 5x_2 = 150$$

The fact that non-negative number of kg of each food must be purchased produces the constraints:

$$x_1 \geq 0$$
$$x_2 \geq 0$$

Since the cost of beans food, f_1 and plantain food, f_2 is N3 and N5 respectfully, per kg. Then, the objective function of the dietary requirement problem, which is to be minimized, is

$$f(x_1, x_2) = 3x_1 + 5x_2$$

subject to:
$$3x1 + 4x2 \geq 340$$
$$x1 + x2 \geq 100$$
$$x1 + 5x2 \geq 150$$
$$x1, x2 \geq 0$$

And this gives the mathematical models of the dietary requirements, i.e. the linear system of the constraints. This can solved either algebraically or graphically.

The result of the solution to the above system of constraint indicates that the optimal solution to the problem is $f(x_1, x_2) = 3x_1 = 100 + 5x_2 = 10) = 350$. That is, in order to minimize cost and still achieve the objective of meeting the dietary requirements. 100kg of food f_1 and 10kg of food f_2 should be purchased

5. Allocation or Transportation Problems

Allocation or Transportation problems are used interchangeably in this context, and they represent, by far, the commonest kind of business problems, especially if we accept the view that investment is best considered as a problem allocating resources over time. Every business is concerned with making good use of scarce resources-ensuring that at every point in time each man or machine is doing the right job .to quote lord Robin's famous definition of economic, as concerned with the relationship between ends and scarce means which have alternative user". The emphasis of Robin's definition of economics is that if there is to be an economic problem (or allocation problem in business) the resources in question must

- Be scarce
- Have alternative uses

Unless both of these conditions are met or satisfied there is no allocation problem. If resource are not scare, they can used in whatever quantities that are necessary to do the job at hand. And, unless resources have alternative uses, there is no possibility of allocating them in the most useful way; and so, no decision to take.

Over the years, business has developed various devices like the single bar chart for determining how the scarce resources can be used during any period in time. Note that it is not simply a matter of only resources being in limited quantity; time is also limited (scare). Thus allocation means allocating the scares resources to do the best job that they can in a given period of time.

One of the techniques, according to Hillier and Lieberman (1980), in solving allocation problem in business is the use of linear programming. These techniques can be used if the relationship between the variable in the problem are linear. This means that the function (equations) linking these variables are straight-line ones. This paper, however, examines the basic aspects and concepts of linear programming as a practical approach towards solving transportation cost problems. The essence of the solution to these problems is the optimization of some function, called the objectives function' - that is, what a firm aims at. The firm normally wishes to maximizes or minimize this function subject to certain constraints, which are also linear, and which can also be expressed as linear function of the relevant variables.

The mathematical structure of the transportation problems can be stated thus:

Minimize
$$\sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij}X_{ij}$$

Subject to
$$\sum_{i=1}^{n} X_{ij} = a \; (i = 1, 2, \dots, m)$$

$$\sum_{j=1}^{m} X_{ij} = b \; (j = 1, 2, \dots, n)$$

Such that
$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j$$

Where m = number of supply sources
n = number of demand destinations
a = number of units from the supply source
b = number of units required by the demand destination
X_{ij} = number of units transported
C_{ij} = cost of units transported

6. The Transportation Matrix

Let there be a 4x4 transportation matrix where f_1, f_2, f_3 and f_4 represent four factories with supplies s_1, s_2, s_3 and s_4 respectively. Let there be, also, be four warehouses: w_1, w_2, w_3 and w_4 with demand requirements d_1, d_2, d_3 and d_4 respectively. If the unit of cost of transporting an item X_{ij} from factory f_i to warehouse w_j is given by C_{ij}, then the transportation matrix can be represented as shown below:

The Transportation Matrix

Factories	w_j / f_i	Warehouses				Factory supply, s_i
		w_1	w_2	w_3	w_4	
	f_1	X_{11}	X_{12}	X_{13}	X_{14}	s_1
	f_2	X_{21}	X_{21}	X_{23}	X_{24}	s_2
	f_3	X_{31}	X_{32}	X_{33}	X_{34}	s_3
	f_4	X_{41}	X_{42}	X_{43}	X_{44}	s_4
Warehouse demand (Requirements), d_j		d_1	d_2	d_3	d_4	

While X_{11} in the matrix denote an item from factory f_1 to warehouse w_1 and X_{44} represent an item from factory f_4 to warehouse w_4; C_{11} represent the cost of transportation X_{11} and C_{44}, the cost of transporting X_{44}. It is expected that, for a balanced problem, the total supplies of all the factories, ($\sum a_i$) and the total demand requirements of the warehouse, b_i) should be equal.

The unit cost $C_{ij's}$ are arranged in such a way that they corresponds to the rows of the factories and the columns of the warehouses. These costs must be proportional to the distances from the supply sources to the demand destination. And for optimal results, all factories supplies must be exhausted and the requirement must be satisfied.

Note:

- Any of the transportation techniques that are used to solve a particular problem must be carefully applied and it must be fully completed before trying another one.
- The transportation matrix is very significant in the solution of a transportation problem. It has to be carefully drawn to actually represent the problem data. Any misrepresentation of data will not lead to the achievement of the desired objective of minimization of transportation cost.

The Simplex Procedure for the Transportation Problem: Since transportation problem is basically a linear programming problem. Albeit a special one, it can be solved by applying the Simplex Procedure. However, because of its special nature, tremendous time consuming and computational effort can be saved by adopting some short-cut procedure of the streamlined transportation simplex method.

The Streamlined Transportation Simplex Method: In a standard linear programming problem the number of basic variable in a feasible solution is given by the number of constraints. However, because of the special features of the standard transportation problem, the number of decision variable is equal to the product of m (origins) and n (destination). Although, there are (m + n) functional constraints, the number of basic variable (i.e. those variables is limited to (m + n - 1). This is because they are equality constraints, and the set (m + n) has one extra or redundant equation that can be deleted without loss of information. In other words, any of the functional constraints can be satisfied whenever the other (m + n) constraints are satisfied.

A linear programming problem having n - variable and m - functional constraints, where n is greater than m (i.e. n > m); a basic solution to the problem may be obtained by setting n – m variable equal to zero, and solving the m constraints equation for the remaining n variables. The n - m variables that are set equal to zero are referred to as **non-basic** variables while the remaining n variables (usually non zeros) are the **basic variables**.

A basic solution need not be a feasible solution, for instance, a basic solution may not satisfy the non-negative condition. However, when a basic solution is feasible (i.e. when the entire basic variables are non-negative), it referred to as a **basic feasible solution**.

An important feature of the transportation problem is that the data should put in the form of a rectangular tableau, with the input data – C_{ij}, d_j and s_i

7. Determinant of Initial Basic Feasible Solution

A number of criteria have been put in place for the selection of the variable in the initial feasible solution. Five of the most frequently used are the:

- Northwest Corner Rule
- Least Cost Rule
- Vogel's Approximation Method
- Russel's Approximation Method
- Stopping Stone Rule

This paper has considered for an illustration, only the Northwest Corner Rule.

Suppose that a conglomerate of cement companies have come under considerable pressure to eliminate its discriminatory allocation/distribution practices. Company officials have reached an agreement with their customers that, during the next five years allocation / distribution of cement (in 50 Kg bags) to warehouses at Abuja, Lokoja, Ogoja and Suleja would come from the following cement factories: Calabar, Gboko and Port Harcourt. The transportation cost (in Naira) per 50 Kg bags of cement from one factory to a particular warehouse (C_{ij}), the demand requirements (in bags cement - d_j) of the four warehouses and the available supply (in bags of cement - s_j) from the three factories are as presented in the table below:

Cement allocation

Factories	Destination / Source	Warehouses				Factory supply, s_i
		Abuja	Lokoja	Ogoja	Suleja	
	Calabar	30	70	60	40	500
	Gboko	20	40	30	20	200
	Port Harcourt	40	30	80	50	300
Demand (Requirements), d_j		300	300	200	200	

Beside the input data, the only information needed by the transportation simplex method is the current basic feasible solution, the current values of U_i, V_j and the resulting values of ($C_{ij} – U_i - V_j$) for the non-basic variables, $X_{ij's}$; where U_i is the i^{th} row, V_j is the j^{th} column, X_{ij} is the item in the i^{th} row and column and C_{ij} is the cost associated with X_{ij}

8. Steps involved in constructing an initial basic feasible solution

Begin: All source and destination column of the transportation simplex tableau are initially under consideration for providing a basic variable (allocation)

Step 1: From amongst these row and column select and circled the first basic variable according to some criteria as specified below

Step 2: To the variable selected make an allocation large enough to use up all the available supply in its row or all the unsatisfied demand in its column, whichever that is smaller

Step 3: Eliminate the row or column (whichever had the smaller outstanding supply or demand) from consideration for further allocation. If the row and column, both, have the same 'zero' supply and demand outstanding, then select the row rather than the column as the one to be eliminated leaving the column; this is often helpful in providing a degenerate basic variable, if needed. A degenerate basic variable has a circled allocation of zero.

Step 4: Repeat steps 1, 2, 3 for the row and columns remaining, until only one row or column is remaining in the simplex tableau.

Step 5: When only one row or column remains under consideration, then the procedure is completed by selecting every remaining variable associated with the row or column to be basic variable with the only feasible allocation.

9. Solution to the illustration using the Northwest Corner Rule

This rule generates a feasible solution not more than (m + n – 1) positive values. The variables which occupy the northwest corner position in the transportation tableau are chosen as the basic variable. Thus X_{11} is selected. Therefore, if X_{ij} was the last basic variable selected, and then next select $X_{i, j+1}$ if the source still has any supply remaining.

$X_{i, j+1}$ implies that, along the same row but the next adjacent column; otherwise select $X_{i+1, j}$ next, that is, the next row below it, but on the same column.

Considering the data on cement allocation tableau, m = 3 and n = 4, the procedure should find an initial basic feasible solution having (m + n + 1), i.e. (3 + 4 – 1 = 6) basic variables. The first allocation is made to X_{11} and equals 300 bags. This exactly satisfies the demand requirement in column 1 and eliminates this column from further consideration. Thus, we are left with a supply of 200 bags remaining in row 1. Thus, we select $X_{1, 1+1} = X_{12}$ to be the next basic variable. Since the demand requirement of this column is larger than the supply remaining, all the 200 bags are allocated to X_{12}. The supply in row 1 is now exhausted, and the row is therefore eliminated from further consideration.

With row 1 and column 1 now eliminated, $X_{1+1, 2} = X_{22}$ becomes the variable occupying the northwest corner of the tableau, and it is selected as the next basic variable. At this point, the remaining demand of column 2 is 100 bags. This is satisfied by allocating 100 out of 200 bags of supply available in row 2. With this allocation, the demand of column 2 is fully satisfied, and the column is eliminated and $X_{2, 2+1} = X_{23}$ is selected as the next basic variable. The procedure continues until initial basic feasible solution is obtained as demonstrated in the table below.

The arrows have only been added for a better understanding, and to show the order in which the basic variables were selected and the allocation made at each stage.

Cement allocation

	Destination / Source	Warehouse: Abuja	Lokoja	Ogoja	Suleja	Factory Supply, s_j
Factories	Calabar	30 (300) →	70 (200) ↓	60	40	500
	Gboko	20	40 (100) →	30 (100) ↓	20	200
	Port Harcourt	40	30	80 (100) →	50 (200)	300
Demand (Requirements), d_j		300	300	200	200	

Thus, we have six basic variables (X_{11}, X_{22}, X_{23}, X_{33} and X_{34}) in the solution. The rows sums are equal the factory warehouse capacities, and the column sum equal to the demand requirements of the destination. Therefore, we have a basic feasible solution to the transportation problem

From the basic feasible solution, the total transportation cost

$$Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij} \text{ is obtained as follows}$$

$$Z = 30X_{11} + 70X_{12} + 40X_{22} + 30X_{23} + 80X_{33} + 50X_{34}$$
$$= 30(300) + 70(200) + 40(100) + 30(100) + 80(100) + 50(200)$$
$$= N48,000$$

The initial basic solution provided by the northwest corner role may be far from optimal, if this method provides an optimal solution, that result may be purely fortuitous; this is because the northwest corner rule completely ignores the route differences in transportation cost, and therefore makes no conscious attempt at choosing the lowest cost route. The other four criteria take into account the differences in the magnitude of transportation cost.

10. Conclusion

Allocation or transportation cost problems could be solved more easily with linear programming approach provided that one s able to formulate the problem using linear equation for both the objective function and the binding constraints or restrictions. In simple diagrams, the optimal solution would be on the boundary of the feasible area; or more specifically, the optimal solution would occur at the vertex of the polyhedral feasible set. However, one would not be able to represent complex linear programming problems on two-dimensional diagram; other approaches like the 'simplex method' are available to tackle such problems.

Linear programming, as has been discussed in relation to allocation or transportation cost problems, is of immense importance, and very vital decision-making process. It has impacted significantly and positively towards solving allocation and transportation cost problems. Production and marketing managers are employed to adopt the linear programming approach as the best option of transportation finished goods or products to their destination (warehouse)

Linear programming can also aid production and marketing managers in performing their daily tasks of production mix, labour and scheduling, job assignment and allocation, marketing research, media selection, ingredient mix, financial portfolio, etc.

11. References

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